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# On the Use of Aboodh Transform for Solving Non-integer Order Dynamical Systems

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**Abstract:** In this research study, a transform technique called the Aboodh transform has been employed to find closed form solutions of initial value problems in non-integer order ordinary differential equations. A newly derived theorem in the present study provides a general structure for finding the exact solutions of such non-integer order problems to be solved by this transform technique. Upon comparison with existing transforms including Laplace, Mellin, Sumudu and Elzaki, the Aboodh transform can be found in good agreement. Various non-integer order differential equations including Bagley-Torvik type equation have successfully been solved by the Aboodh transform technique.

**Keywords:** Fractional calculus, Integral transform, Caputo derivative, Bagley-Torvik.

## 1. INTRODUCTION

Leibniz, in one of his letters to L'Hopital, wrote on 30 September 1695 that one day very useful results will emerge from the interpretation of semi-derivative of a mathematical function. This date is considered to be the exact birthday of what is known today the Fractional Calculus (FC) or the calculus of non-integer order differentiation and integration. Nowadays, various mathematical models having non-integer order derivatives and/or integrals have naturally been encountered by mathematicians, physicists and engineers in their research studies where it has been observed that such models are much more accurate and efficient than the models in the classical (integer-order) calculus and it is due to the fact that such operators (derivative and integral) describe memory and hereditary properties of various real materials and physical and biological processes whereas such properties are entirely neglected in the standard calculus (Yusuf, *et al.*, 2018), (Sun, *et al.*, 2018), (Rosa, and Torres, 2018), (Povstenko, and Kyrylych, 2018) and (Baleanu, 2012). Thus, the field of FC is now more frequently being used by various engineers and applied mathematicians. Applications of this interesting fields are ubiquitous in many areas including diffusion process, wave propagation, fluid dynamics, quantum mechanics, particle physics, bio-engineering, mechanical, electrical and civil engineering, to name just a few (Abro, *et al.*, 2018), (Tarasov, 2011).(Dalir, and Bashour, 2010), (Sabatier, 2007), (Magin, 2006), (Shivanian, and Jafarabadi, 2000). Most of the mathematical models arising in such fields are based upon ordinary differential equations having non-integer order derivatives prescribed with some condition(s) and required to be treated with reliable and accurate techniques to determine qualitative and quantitative behavior of their possible existing solutions. In order to determine closed form solutions of the models, various integral transform techniques are in use including the most celebrated Laplace transform technique. Other techniques include Mellin, Sumudu, Natural, and Elzaki with a few more recently proposed. Various types of special functions form the very basics of non-integer order calculus including the most commonly used special function in fractional differential equations known as the Mittag-Leffler function (Górska, *et al.*, 2018).

In an attempt to find the closed form solutions of classical ordinary differential equations, the author in (Aboodh, 2013) proposed a new integral transform and successfully applied it for obtaining the required solutions. Recently, many of its properties and applications have been derived in (Ahmad and Tariq, 2018) where in the authors have extensively discussed the connection of this transform with some existing well-known transforms. In this continuation, various first and higher order ordinary differential equations were solved in (Ahmad, and Tariq, 2018) and (Aboodh, *et al.*, 2018).In recent times, an interesting application related to population dynamics of growth and decay problems is dealt with the Aboodh transform in

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(Aggarwal, *et al.*,2018). However, all the mathematical models in such studies are solved from the domain of classical (integer-order) calculus wherein one has to deal with ordinary differential equations possessing only integer-order derivatives.

The fundamental aim of the present paper is to propose a mechanism to solve non-integer order initial value problems with the Aboodh transform technique. In this regard, a new theorem with its proof has been presented in the section **3** preceded by section **2** which provides some basic concepts related to non-integer order calculus and the Aboodh transform technique itself. Section **4** presents some results and discussions about the numerical experiments including the Bagley-Torvik model to be solved by the transform technique under consideration. Finally, some concluding remarks are provided in the section **5**.

## 2. MATHEMATICAL PRELIMINARIES

For rest of the analysis in the paper, it is important to be familiar with some fundamental concepts of non-integer order calculus and the Aboodh transform technique. Therefore, few important definitions and theorems are presented next.

**Definition 2.1.** (Podlubny, 1999) The Cauchy formula for repeated integration in classical calculus when generalized for an arbitrary real number $\xi > 0$ gives the definition for finding the integration of $y(t)$ for order $\xi > 0$. This modified form of integration is called the Riemann-Liouville fractional integral of order $\xi > 0$:

$$J^{\xi}{}_{0,t}\left[y(t)\right] = \frac{1}{\Gamma(\xi)}\int_0^t y(\tau)(t-\tau)^{\xi-1}\,d\tau,$$

where $t > 0$, $\Gamma(\cdot)$ is the Euler's Gamma function.

**Definition 2.2.** (Podlubny, 1999) The non-integer order derivative of function $y(t)$ called Riemann-Liouville derivative of order $\xi > 0$ is defined by:

$$^{RL}D^{\xi}{}_{0,t}\left[y(t)\right] = \frac{1}{\Gamma(m-\xi)}\frac{d^m}{dt^m}\int_0^t y(\tau)(t-\tau)^{\xi-1}\,d\tau,$$

where $t > 0$, $(m-1) \le \xi < m \in \mathrm{N}\cdot$

**Definition 2.3.** (Podlubny, 1999) The non-integer order derivative of function $y(t)$ called the Caputo derivative of order $\xi > 0$ is defined by:

$$^{C}D^{\xi}{}_{0,t}\left[y(t)\right] = \frac{1}{\Gamma(m-\xi)}\int_0^t y^{(m)}(\tau)(t-\tau)^{m-\xi-1}\,d\tau,$$

where $t > 0$, $(m-1) < \xi \le m \in \mathrm{N}\cdot$

**Note.** The Caputo definition is preferred over the Riemann-Liouville definition due to the reasons of availability of integer order conditions having classical physical interpretations whereas the latter requires to have non-integer order derivatives conditions for which neither physical nor geometrical have been proposed yet for universal acceptance.

**Definition 2.4.** (Górska, *et al.*, 2018). The Aboodh transform for functions of exponential order is defined by the following integral equation:

$$Y(p) = A\left[y(t)\right] = \frac{1}{p}\int_0^{\infty}\exp(-pt)\,y(t)\,dt,$$

where $t \ge 0$ and $k_1 \le p \le k_2\cdot$ Moreover, the functions should belong to the following set in order to have the Aboodh transform to exist and for convergence of the integral above:

$$A = \left\{y(t) : \exists\, M, k_1, k_2 > 0, \left|y(t)\right| < M\exp(-pt)\right\}\cdot$$

**Theorem 2.1.** (Aboodh, 2013) If $A\left[y(t)\right] = Y(p)$ then Aboodh transform for integer-order derivatives of $y(t)$ is determined as follows:

$$A\left[y^{(n)}(t)\right] = p^n Y(p) - \sum_{i=0}^{n-1}\frac{y^{(n)}(0)}{p^{i+2-n}}\cdot$$

**Lemma 2.1.** (Podlubny, 1999) The non-integer order derivative operator called the Caputo fractional operator is related to the non-integer order Riemann-Liouville integral by the following identity:

$$^{C}D^{\xi}{}_{0,t}\,y(t) = J^{m-\xi}{}_{0,t}\left(D^m y(t)\right),$$

where $(m-1) < \xi < m \in \mathrm{Z}^+\cdot$

## 3. MATERIAL AND METHODS

In this section, a new theorem has been proposed to find the Aboodh transform for the non-integer order derivatives of function $y(t)$ which, in turn, becomes very useful to solve various non-integer order initial value problems under the Caputo's type ordinary differential equations.

### 3.1. Aboodh Transform Technique

The transform technique called the Aboodh Transform is an integral transform which has successfully been tested on various integer-order ordinary and partial differential equations, for example see in (Aggarwal, *et al.*, 2018). and the references therein. It was first proposed in (Górska, *et al.*, 2018). and later analyzed in detail by many authors. It has very interesting properties and relations with existing transform techniques including Laplace, Mellin, Sumudu, Natural and Elzaki transform. Nevertheless, all of these research studies explored the efficiency of the Aboodh transform to solve integer-order differential equations.

In this present research study, the Aboodh transform is being tested on non-integer order ordinary differential equations subject to some initial condition(s). In this connection, a new theorem has been proposed in the present study whose proof given below helps us to get a general mechanism to determine the closed form solutions of such non-integer order equations.

**Theorem 3.1.** If $Y(p)$ is the Aboodh transform of a function $y(t)$ then the Aboodh transform of non-integer order derivatives for $y(t)$ under the Caputo operator having order $\xi > 0$ is proposed as follows:

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right]$$

$$= p^{\xi}Y(p) - \sum_{i=0}^{n-1}\frac{y^{(i)}(0)}{p^{i+2-\xi}}\cdot$$

**Proof.** The above Lemma 1 can be used to derive the required result. From this lemma, one obtains

$${}^{C}D^{\xi}{}_{0,t}y(t) = J^{n-\xi}{}_{0,t}\left(D^{n}y(t)\right),$$

where $n \in \mathrm{N}\cdot$

Let $D^{n}y(t) = k(t)$, then one has

$${}^{C}D^{\xi}{}_{0,t}y(t) = J^{n-\xi}{}_{0,t}\left(k(t)\right)\cdot$$

Taking the Aboodh transform on both sides, one obtains

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right] = A\left[J^{n-\xi}{}_{0,t}\left(k(t)\right)\right]\cdot$$

The Aboodh transform for integer-order integrals can be written as

$$A\left[I^{n}(t)\right] = \frac{1}{p^{n}}Y(p),$$

where $I^{n}(t) = \int_{0}^{t}\cdots\int_{0}^{t} y(t)(dt)^{n}\cdot$ Thus,

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right] = \frac{1}{p^{n-\xi}}A\left[k(t)\right]$$

$$= \frac{1}{p^{n-\xi}}\left[p^{n}Y(p) - \sum_{i=0}^{n-1}\frac{y^{(i)}(0)}{p^{i+2-n}}\right]$$

Hence, further simplification yields the following:

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right] = p^{\xi}Y(p) - \sum_{i=0}^{n-1}\frac{y^{(i)}(0)}{p^{i+2-\xi}}\cdot$$

This completes the required proof for the Aboodh transform for non-integer order derivatives under the Caputo operator.

## 4. RESULTS AND DISCUSSIONS

This section is devoted to test the above proposed theorem on some non-integer order differential equations. Three such equations are solved including the Bagley-Torvik equation. Aboodh transform when applied on these equations yields the closed form solutions which are in good agreement with those obtained through other well-known existing transforms.

**Example 1.** (Diethelm, 1997). Consider the following linear non-integer order in-homogeneous initial value problem:

$${}^{C}D^{\xi}{}_{0,t}y(t) + y(t) = \frac{2t^{2-\xi}}{\Gamma(3-\xi)} + t^{2},\ y(0) = 0,$$

where $\xi \in (0,1)\cdot$

Taking Aboodh transform on both sides, one obtains

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right] + A\left[y(t)\right] = A\left[\frac{2t^{2-\xi}}{\Gamma(3-\xi)} + t^{2}\right]\cdot$$

Using some basic properties of the Aboodh transform and the proposed theorem **3.1**, one obtains

$$p^{\xi}Y(p) - p^{\xi-2}y(0) + Y(p)$$

$$= \frac{2}{\Gamma(3-\xi)}\left(\frac{\Gamma(3-\xi)}{p^{4-\xi}}\right) + \frac{2}{p^{4}},$$

$$Y(p)\left(1+p^{\xi}\right)=\frac{2p^{-\xi}\left(1+p^{\xi}\right)}{p^{4-\xi}},$$

$$Y(p)=\frac{2}{p^{4}}\cdot$$

Inverse Aboodh transform yields the following closed form solution: $y(t)=t^{2}\cdot$

**Example 2. (**Podlubny, 1999). Consider the following linear non-integer order in-homogeneous initial value problem of Bagley-Torvik type:

$$D^{2}y(t)+{}^{C}D^{\xi}{}_{0,t}y(t)+y(t)=t+1,$$

where $y(0)=y'(0)=1$ and $\xi=1.5\cdot$

Taking Aboodh transform on both sides, one obtains

$$A\left[D^{2}y(t)\right]+A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right]+A\left[y(t)\right]$$
$$=A\left[t+1\right]\cdot$$

Using some basic properties of the Aboodh transform and the proposed theorem **3.1**, one obtains

$$p^{2}Y(p)-\frac{1}{p}-1+p^{\xi}Y(p)$$
$$-\left(\frac{1}{p^{2-\xi}}+\frac{1}{p^{3-\xi}}\right)+Y(p)=\frac{1}{p^{3}}+\frac{1}{p^{2}},$$
$$Y(p)\left(p^{2}+p^{\xi}+1\right)=\frac{1}{p^{2}}\left(1+\frac{1}{p}\right)\left(p^{2}+p^{\xi}+1\right),$$
$$Y(p)=\frac{1}{p^{2}}\left(1+\frac{1}{p}\right)\cdot$$

Inverse Aboodh transform yields the following closed form solution: $y(t)=1+t\cdot$

**Example 3.** (Diethelm, 1997) Consider the following linear non-integer order problem with $y(t)$ being a smooth function:

$${}^{C}D^{\xi}{}_{0,t}y(t)$$
$$=\begin{cases}\dfrac{2}{\Gamma(3-\xi)}t^{2-\xi}-y(t)+t^{2}-t, & \xi>1,\\ \dfrac{2}{\Gamma(3-\xi)}t^{2-\xi}-\dfrac{1}{\Gamma(2-\xi)}t^{1-\xi}-y(t)+t^{2}-t, & \xi\le 1\cdot\end{cases}$$

The initial conditions are taken to be $y(0)=0$ for $\xi\le 1$ and $y'(0)=-1$ for $\xi>1$.

The problem can be solved under both of the given conditions. Let's take the second case and take the Aboodh transform on both sides:

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right]$$
$$=A\left[\frac{2}{\Gamma(3-\xi)}t^{2-\xi}-\frac{1}{\Gamma(2-\xi)}t^{1-\xi}-y(t)+t^{2}-t\right],$$

where $\xi\le 1$.

Once again, the basic properties of the Aboodh transform and application of the proposed theorem yield the following simplifications

$$A\left[{}^{C}D^{\xi}{}_{0,t}y(t)\right]$$
$$=A\left[\frac{2}{\Gamma(3-\xi)}t^{2-\xi}-\frac{1}{\Gamma(2-\xi)}t^{1-\xi}-y(t)+t^{2}-t\right],$$
$$\left(1+p^{\xi}\right)Y(p)$$
$$=\frac{2}{\Gamma(3-\xi)}\left(\frac{\Gamma(3-\xi)}{p^{4-\xi}}\right)$$
$$-\frac{1}{\Gamma(2-\xi)}\left(\frac{\Gamma(2-\xi)}{p^{3-\xi}}\right)+\frac{2}{p^{4}}-\frac{1}{p^{3}},$$
$$\left(1+p^{\xi}\right)Y(p)=2p^{\xi-4}-p^{\xi-3}+\frac{2}{p^{4}}-\frac{1}{p^{3}},$$
$$Y(p)=\frac{2}{p^{4}}-\frac{1}{p^{3}}\cdot$$

Inverse Aboodh transform yields the following closed form solution: $y(t)=t^{2}-t\cdot$

Thus, the Aboodh integral transform can now be used to solve any linear non-integer order ordinary differential equation using some fundamental properties derived for integer order derivatives in the relevant literature and the above proposed theorem derived for non-integer order derivatives under the Caputo's type operator.

## 5. CONCLUSION

In the present research study, a newly devised integral transform not tested before in literature on non-integer order differential equations has successfully been employed to get the closed form solutions of some non-integer order initial value problems under Caputo's type operator. Another contribution of the present work is the proof of a theorem which ultimately helps to get such solutions. This theorem is provided while keeping in mind the fractional nature of the problems under consideration. Later, the application of this proposed theorem along with some fundamental properties of the Aboodh transform was employed to get the exact closed form solutions of some differential equations taken from the relevant literature including the Bagley-Torvik non-integer order differential equation. The results obtained can be compared with some well-known integral transform techniques such as the Laplace, Mellin, Sumudu, Natural and Elzaki transform. All the results found in the present research work are in good agreement with other transform techniques as mentioned.

**Conflict of Interest**

The authors declare that they have no conflict of interest.

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**Authors' Contribution**

All the authors have contributed equally towards the completion of this research work and they have approved the final submitted version of the paper.

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