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Development of Improved Scheme for Numerical Integration of Autonomous and Non-Autonomous Initial Value Problems

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Abstract: In this research study, an explicit and single stepnew third order improved scheme for numerical integration of autonomous and non-autonomous initial value problemsalso called Cauchy problems in ordinary differential equations has been developed. Linear stability analysis with corresponding stability region is drawn and error analysis has been provided to confirm third order accuracy of the improved scheme. Inclusion of a partial derivative with respect to the dependent variable with in two slopes of the improved scheme has improved its efficiency in terms of local and global truncation errors. Finally, numerical examples are provided to show performance of the improved scheme in comparison with other existing methods having same order of local accuracy. Throughout the study it has been observed from the results and comparison of the improved scheme that develop algorithm is superior to some existing numerical methods of same order in term of error analysis as well as accuracy viewpoint.The software MATLAB is used to justify the results and graphical representation of the improved scheme.

Keywords: Runge-Kutta, Stability, Local Truncation Error, Non-autonomous, Cauchy problems.

1. INTRODUCTION

Ordinary Differential equations usually express all-natural phenomenon, which come across in this universal system. Their usage is present everywhere in science, engineering, economics as well as in social sciences to name a few as (Lambert, 1973). (Dory, 1989): Few ordinary differential equations with initial value problems are autonomous and non-autonomousby nature, and solution of suchODE's with IVP'S can be obtained by analytic and numerical schemes, but analytic schemes mostly are unable to find their solution, so we have to refer numerical schemesas described in (Butcher, 2016). Many researchers are familiar with such ODE'ssince long time and they have formulated numerous schemes for thesolutions of suchODE's asshown in(Owolanke, *et al.*,2017). (Mukaddes*et al.*,2016) (Ashiribo*et al.*,2013). These schemes are single and multi-step schemes.The single step schemes mainly include Euler's method, Modified and Improved methods, Explicit and Implicit Runge Kutta method.Whereas multi step schemes include Adams interpolation and external interpolation formula, prediction and correction formula, that methods can be utilized for differential equation as well as system of differential equation. Thus, numerical methods are very important part for estimating solution of ordinary differential equations, which could not be ignored.

2. DERIVATION OF THE IMPROVED SCHEME

To illustrate various numerical methods for the solution of ordinary differential equations. We consider the general first order ordinary differential equation with an initial condition, also called Cauchy Problem, as given below:

$$\frac{dy}{dx} = f(x_n, y_n), \quad y(x_0) = y_0 \qquad (1)$$

Existence of unique solution of (1) is assumed for the integration interval of $x \in [x_0, x_n]$. Here exact solution is denoted by $y(x_n)$ whereas the numerical solution is by y_n, taking the step size $h = \frac{x_n - x_0}{N}$ where $N = 1,2,3,\ldots$

Generally, in n-dimensional real space y, y_0, y_n and f are regarded as vectors, which are sought by integrating (1) from x_0 to $x_0 + h$ in the form

$$y_{n+1} = y_n + \int_{x_0}^{x_0+h} f(x_n + y_n)\, dx$$

or, in equivalent form

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$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$$

We set different s,such as 1,2,3. First we set $s=1$, then above form can be written as

$$y_{n+1} = y_n + h(b_1k_1)$$

In similar manner, we set $s=2$

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2)$$

$s=3$, then above form becomes

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3) \qquad (2)$$

Where k_1, k_2 and k_3 are the slopes, determined by

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + a_2h, y_n + h(b_{21}k_1) + h^2(c_{21}k_1)f_y\right)$$
$$k_3 = \left(x_n + a_3h, y_n + h(b_{31}k_1 + b_{32}k_2) + h^2(c_{31}k_1)f_y\right)$$

The value of k_2 and k_3 are expanded by Taylor series. The Taylor series expansion of $Y(x_n, y_n)$ is

$$Y(x_n, y_n) = y(x) + hf + h^2\left[\frac{1}{2}f_x + \frac{1}{2}f_y f\right] + h^3\left[\frac{1}{6}f_{xx} + \frac{1}{3}f_{xy}f + \frac{1}{6}f_{yy}f^2 + \frac{1}{6}f_y^2 f + \frac{1}{6}f_x f_y\right]$$
$$+h^4\left[\begin{matrix}\frac{1}{24}f_{xxx} + \frac{1}{8}f_{xxy}f + \frac{1}{8}f_{xyy}f^2 + \frac{5}{24}f_y ff_{xy} + \frac{1}{8}f_x f_{xy} + \frac{1}{24}f_{yyy}f^3 + \\ \frac{1}{6}f_y f^2 f_{yy} + \frac{1}{8}f_x ff_{yy} + \frac{1}{24}f_y^3 f + \frac{1}{24}f_x f_y^2 + \frac{1}{24}f_y f_{xx}\end{matrix}\right] + O\left(h^5\right) \qquad (3)$$

Expanding k_2 and k_3 in Taylor's series, after this we substitute the result of k_1 ,k_2 and k_3 into(2), then equate the coefficients of powers of *h* up to h^3 with that of $(3)$to obtain the following order conditions:

$$b_1 + b_2 + b_3 = 1 \qquad a_2b_3b_{32} = \frac{1}{6} \qquad a_2b_2 + a_3b_3 = \frac{1}{2}$$
$$\frac{1}{2}\left(a_2^2b_2 + a_3^2b_3\right) = \frac{1}{6} \qquad b_2b_{21} + b_3b_{31} + b_3b_{32} = \frac{1}{2} \qquad a_2b_2b_{21} + a_3b_3b_{31} + a_3b_3b_{32} = \frac{1}{3} \qquad (4)$$
$$b_2c_{21} + b_3c_{31} + b_3b_{21}b_{32} = \frac{1}{6} \qquad \frac{1}{2}\left(b_2b_{21}^2 + b_3b_{31}^2 + b_3b_{32}^2\right) + b_3b_{31}b_{32} = \frac{1}{6}$$

This nonlinear system having 8 equations and 10 unknowns.So further we search out that this nonlinear systemhas trivial and non-trivial solution, for this purpose we need to determine parameters.One of the solutions of the above nonlinear system (4) forms a three-stage explicit single step third order improved scheme for numerical integration of autonomous and non-autonomous initial value problems as given below:

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{2}{3}h, y_n + hk_1\left(\frac{2}{3} + hf_y\right)\right)$$
$$k_3 = f\left(x_n + \frac{2}{3}h, y_n - \frac{7}{3}hk_1 + 3hk_2 - 8h^2k_1f_y\right) \qquad (5)$$
$$y_{n+1} = y_n + \frac{h}{12}(3k_1 + 8k_2 + k_3)$$

After getting this new improved scheme(5), we will analyse it for its accuracy, convergence, order of consistency and linear stability. These are the important terms related to an improved scheme for it to be acceptable in the field of computational and applied mathematics as proved in Burden, Richard L., and J. Douglas Faires. (2001)

3. ERROR ANALYSIS

The local truncation error of the proposed improved scheme is defined to be T_{n+1} where

$$T_{n+1} = y(x+h) - y_{n+1}$$

Where $y(x)$ is the theoretical solution and y_{n+1} is approximate solution. Expanded these intoTaylor series about *x* and collecting the terms in *h*, the local truncation error (LTE) of the proposed improved scheme is

$$T_{n+1} = \begin{pmatrix} \frac{1}{27} f_{yyy} f^3 + \frac{1}{81} f_{xyy} f^2 + \frac{2}{9} f_{xy} ff_y + \frac{1}{9} f_x f_{xy} + \frac{1}{18} f_y f_{xx} + \\ \frac{1}{81} f_{xxy} f + \frac{1}{4} f_y^3 f - \frac{5}{18} f_y f^2 f_{yy} + \frac{1}{9} f_x ff_{yy} + \frac{8}{81} f_{xyy} f + \frac{1}{27} f_{xxx} \end{pmatrix} h^4 + O(h^5) \qquad (6)$$

4. CONSISTENCY ANALYSIS

Definition 4.1 Given an initial value problem $y'(x) = f(x_n, y_n);\ y(x_0) = y_0$; an improved scheme with anincrement function $\Phi(x_n, y_n; h)$ is said to be consistent, if

$$\lim_{h\to 0} \Phi(x_n, y_n; h) = f(x_n, y_n)$$

The increment function of the proposed improved scheme is

$$\Phi(x_n, y_n; h) = \frac{1}{12}(3k_1 + 8k_2 + k_3)$$

For consistency, we take $\lim_{h\to 0}$ on both side

$$\lim_{h\to 0} \Phi(x_n, y_n; h) = \frac{1}{12} \lim_{h\to 0} (3k_1 + 8k_2 + k_3)$$

$$= \frac{1}{12} \lim_{h\to 0} \begin{pmatrix} 3f(x_n, y_n) + 8f\left(x_n + \frac{2}{3}h, y_n + hk_1\left(\frac{2}{3} + hf_y\right)\right) + \\ f\left(x_n + \frac{2}{3}h, y_n - \frac{7}{3}hk_1 + 3hk_2 - 8h^2 k_1 f_y\right) \end{pmatrix}$$

$$= f(x_n, y_n)$$

Thus, the proposed improved scheme is shown to be **consistent** with at least **third order accuracy**.

5. LINEAR STABILITY ANALYSIS

To check stability of the improved schemes, we consider Dahlquist's test problem of the form

$$\frac{dy}{dx} = \lambda y(x); \quad y(0) = y_0, \quad \lambda \in C$$

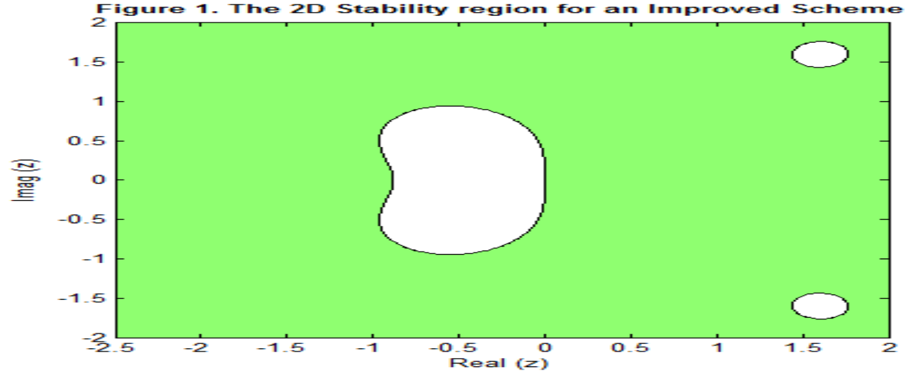
Employing the proposed improved scheme (5)on this test problem, we obtain the following stability function whose linear stability region is shown by the unshaded region in the **(Fig. 1).**

$$k_1 = \lambda y_n;\ k_2 = \lambda y_n\left[1 + \frac{2}{3}h\lambda + h^2\lambda^2\right];\ k_3 = \lambda y_n\left[1 + \frac{2}{3}h\lambda - 6h^2\lambda^2 + 3h^3\lambda^3\right]$$

Substituting all of these values in (5) , the stability function is found to be of the form:

$$R(z) = 1 + z + \frac{z^2}{2} - \frac{z^3}{2} + \frac{z^4}{4} \quad \text{where } z = h\lambda$$

Figure 1. The 2D Stability region for an Improved Scheme



6. NUMERICAL EXPERIMENTS

In this section, some of the linear and nonlinear Cauchy problems in ordinary differential equations have been considered to show the behaviour of the proposed improved scheme against other schemes from well-established literature having same order of accuracy. Absolute maximum error, absolute error at the last nodal point of the given integration interval and CPU values for time have been presented to observe the performance of the developed method in comparison to other methods. Two standard methods called Runge-Kutta Method with Harmonic Mean of Three Quantities (RK3HM) (Abdul 1990) and Runge-Kutta third order method(RK3M)(Butcher, 2016) as shown below have been chosen to compare the numerical results obtained through the newly developed an improved scheme.

Table 1. Errors and CPU values for Cauchy Problem 1

Problem 1. Nonlinear Cauchy problem			
$\frac{dy}{dx}=xy^3-y,$	$y(0)=1,$	$Exact=\frac{2}{\sqrt{2+4x+2e^{2x}}}$	
Step-size /method	**RK3HM**	**RK3M**	**Proposed**
0.1	2.2406e-004	2.1008e-005	1.2688e-005
	2.1363e-004	1.5944e-005	1.1927e-005
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	5.1698e-005	2.5084e-006	7.6936e-007
	4.9489e-005	1.8970e-006	5.0666e-007
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	1.2497e-005	3.0722e-007	6.9878e-008
	1.1988e-005	2.3198e-007	2.0308e-008
	0.0000e+00	0.0000e+00	0.0000e+00
0.0125	3.0773e-006	3.8027e-008	7.5500e-009
	2.9553e-006	2.8697e-008	2.5429e-010
	1.5625e-002	0.0000e+00	0.0000e+00

Figure 2. Absolute Errors for Problem 1 with h=0.1

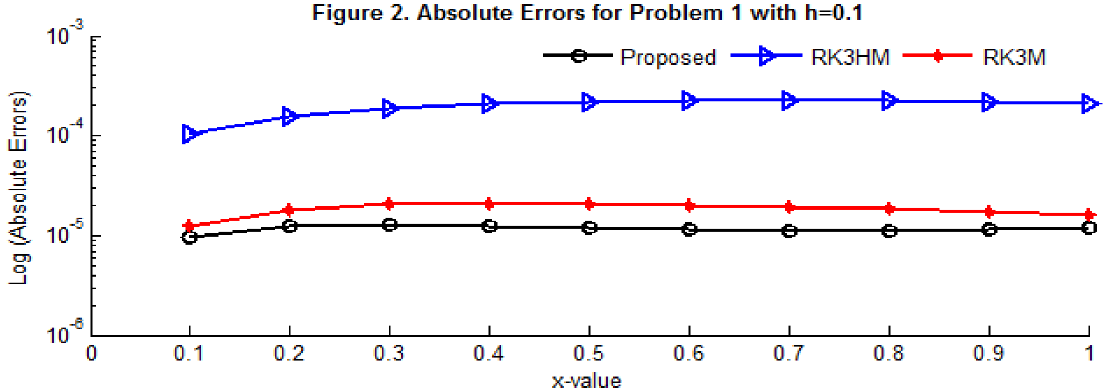


Table 2.Errors and CPU values for Cauchy Problem 2

Problem 2. Nonlinear Cauchy problem $\frac{dy}{dx}=\frac{x^2}{y}$, $y(0)=1$, $Exact=\frac{1}{3\sqrt{9+6x^3}}$			
Step-size /method	**RK3HM**	**RK3M**	**Proposed**
0.1	1.5241e-003	2.2290e-005	1.9548e-005
	1.5241e-003	2.2290e-005	1.9548e-005
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	3.8551e-004	2.8590e-006	1.0562e-006
	3.8551e-004	2.8590e-006	1.0562e-006
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	9.7000e-005	3.6139e-007	6.0505e-008
	9.7000e-005	3.6139e-007	3.3426e-008
	0.0000e+00	0.0000e+00	0.0000e+00
0.0125	2.4332e-005	4.5408e-008	5.9410e-009
	2.4332e-005	4.5408e-008	2.3009e-009
	0.0000e+00	0.0000e+00	0.0000e+00

Figure 3. Absolute Errors for Problem 2 with h=0.1

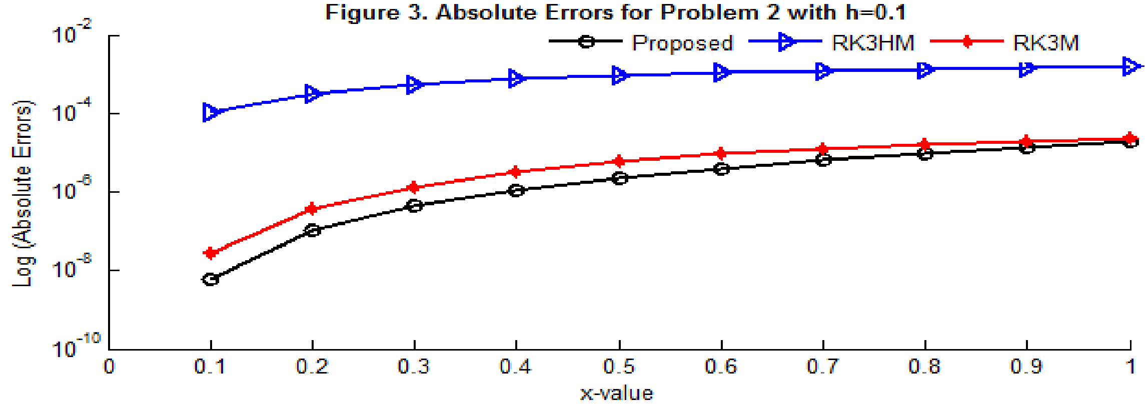
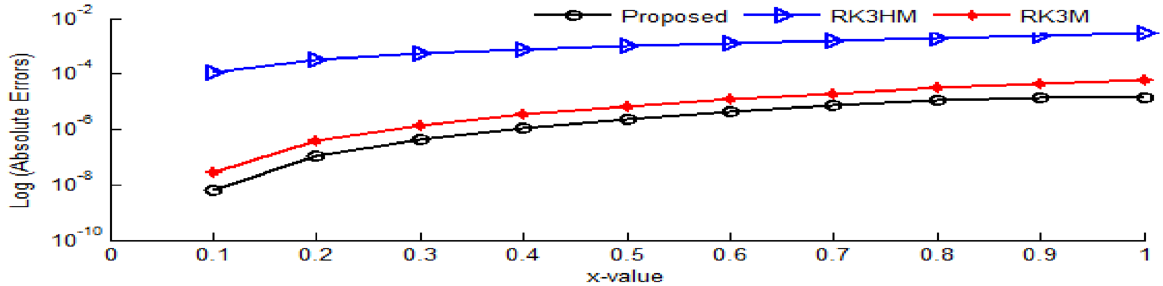


Table 3. Errors and CPU values for Cauchy Problem 3

Problem 3. Nonlinear Cauchy problem $\frac{dy}{dx}=x^2y$, $y(0)=1$, $Exact=e^{\frac{t^3}{3}}$			
Step-size /method	**RK3HM**	**RK3M**	**Proposed**
0.1	3.1171e-003	6.3568e-005	1.4165e-005
	3.1171e-003	6.3568e-005	1.3359e-005
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	8.2411e-004	8.4079e-006	1.6414e-006
	8.2411e-004	8.4079e-006	1.2474e-006
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	2.1194e-004	1.0805e-006	1.9677e-007
	2.1194e-004	1.0805e-006	1.2857e-007
	0.0000e+00	0.0000e+00	0.0000e+00
0.0125	5.3744e-005	1.3694e-007	2.4068e-008
	5.3744e-005	1.3694e-007	1.4347e-008
	0.0000e+00	0.0000e+00	0.0000e+00

Figure 4. Absolute Errors for Problem 3 with h=0.1



7. RESULTS AND DISCUSSIONS

The newly developed third order animproved scheme is capable of solving Cauchy problems in the field ofcomputational and applied mathematics. The maximum error and last error with step sizes 0.1, 0.05, 0.025 and0.0125 are tabulated along-with the values of CPU timing in seconds. One may observe from these tabulateddatathat the absolute maximum and last error produced by the proposed improved scheme are much smallerthan the errors produced by other methods having same order of accuracy while consuming same amount oftime on average.The numerical results obtained throughthe proposed improved scheme produce numerical values approximately close to the exact solution incomparison to the values obtainedthrough Runge-Kutta Method with Harmonic Mean of Three Quantities andRunge-Kutta third order method. Finally, it has been observed that the proposed improved scheme is converging faster than the RK3HM and Runge-Kutta third order method and it is the mosteffective scheme for solving the Cauchy problems in ordinary differential equations as long as it is comparedwith the numerical schemes having same order of local accuracy as that of the proposed improved scheme.

8. CONCLUSION

This paper develops a new single third Order Improved Scheme for Numerical Integration of Cauchy problems in ordinary differential equations. The improved scheme is found to be third order accurate and explicit in nature. Its linear stability analysis gives the stability region which proves conditional stability of the proposed improved scheme. Examples in this paper proved that it is more accurate and effective scheme than some existing standard methods**. (Tables 1 to 3)** above show the maximum error, the last error and CPU times related to all the numerical schemes under consideration for the Cauchy problems with the variation in the step size. In addition, absolute errors produced by the above numerical schemes are smallest in case of the proposed improved scheme as shown by the **(Fig. 2-4).** The computations above evidently display the better accuracy of the improved scheme.

The Runge-Kutta Method with Harmonic Mean grows faster in error than third order Runge-Kutta and the proposed one. Hence, the proposed improved scheme performs best among the numerical schemes taken for comparison. Based on the three Cauchy problems solved above, it follows that the proposed improved scheme is quite efficient specifically in terms of local accuracy. It can be concluded that the proposed improved scheme is powerful and effective in finding numerical solutions of Cauchy type problems arising frequently in the field of computational and applied mathematics.

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