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A Computational Technique for the Solution of Parabolic Type Integro-Differential Equation with a Weakly Singular Kernel

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Abstract: A quartic B-spline collocation method is proposed for numerical solution of partial integro-differential equations with a weakly singular kernel. The scheme is developed by discretization of the time derivative using forward difference formula while the spatial derivative is approximated using quartic B-spline functions. Numerical solution of two test problems are provided to validate the scheme. Accuracy of the method is assessed in terms of L_∞ and L_2 error norms. **Better** accuracy is obtained and the results are also compared with cubic B-spline collocation method.

Keywords: Collocation method, Quartic B-Spline function, Partial integro-differential equation, Weakly singular kernel.

1. INTRODUCTION

Partial integro-differential equations (PIDEs) are widely used in various fields of science and engineering such as fluid dynamics, geophysics, plasma physics, viscoelastic mechanics, civil and aerospace engineering, see (Christensen, 1971,Yoon *et al.*, 2012, Siddiqi and Arshad, 2014, and Ahmad *et al.*, 2015 and the references therein) in order to model several phenomena. Analytical solutions of such equations are not available in general and therefore a numerical solution plays a vital role in study of these types of equations. Consider the following PIDE (Siddiqi and Arshad, 2014):

$$\int_0^\tau \beta(\tau-s)w_\tau(\xi,s)ds - w_{\xi\xi}(\xi,\tau) = f(\xi,\tau), \xi \in [a,b], \tau > 0, \quad (1)$$

subject to the initial condition

$$w(\xi,0) = g_0(\xi), \;\; a \le \xi \le b, \quad (2)$$

and boundary conditions

$$w(a,\tau) = f_0(\tau), w_\xi(a,\tau) = f_2(\tau), w(b,\tau) = f_1(\tau), \tau \ge 0. \quad (3)$$

In Eq. (1), $\beta(\tau)$ is a weakly singular kernel given by

$$\beta(\tau) = \frac{\tau^{(\alpha-1)}}{\Gamma(\alpha)}, 0 < \alpha < 1, \text{at } \tau = 0, \quad (4)$$

where Γ denotes the gamma function.

In Eqs. (1)-(3), $g_0(\xi)$, $f_0(\tau), f_1(\tau)$ and $f_2(\tau)$ are known functions, which are obtained from initial and boundary conditions. The problem (1)-(3) is considered for solution by various authors using different methods like Finite element methods (Yanik and Fairweather, 1988, Lin and Zhang, 1991), finite difference method (Tang, 1993), Cubic B-spline collocation method (Siddiqi and Arshad, 2014).

In this paper a collocation method using B-spline functions is developed for numerical solution of the parabolic type integro-differential equation with the weakly singular kernel (1)-(4).

Several authors have employed the B-spline function techniques for the solution of various types of PDEs and PIDEs and their detail literature can be found in (Saka and Dag, 2008, Saka, *et al.* 2008, Haq, *et al.* 2010, Zhang, 2013, Ali *et al.* 2015 and Ahmad, *et al.* 2015).

2. CONSTRUCTION OF THE PROPOSED METHOD

Suppose the region $R = [a,b] \times [0,T]$ is divided in equal mesh size Δ with the grid points μ_{ij}, where each μ_{ij} is the vertex of the grid point (ξ_i, t_j). In each point $\xi_i = a + ih$ and $\tau_j = jk$, for $i = 0,1,2,3,\ldots,N$ and $j = 0,1,2,3,\ldots,M$, $T = Mk$, h, k are the step sizes in space and time respectively. In Eq. (1), the integral term at $\tau = \tau_{j+1}$ is approximated as (Siddiqi and Arshad, 2014):

$$\int_0^{\tau_{j+1}} \frac{(\tau_{j+1}-s)^{\alpha-1} w_\tau(\xi,s)ds}{\Gamma(\alpha)} = \int_{t_0}^{t_j} \frac{(w(\xi,\tau_1)-w(\xi,\tau_0))(\tau_{j+1}-s)^{\alpha-1}ds}{k\Gamma(\alpha)} + \sum_{m=1}^{j} \int_{\tau_m}^{\tau_{m+1}} \frac{(w(\xi,\tau_{m+1})-w(\xi,\tau_{m-1}))(\tau_{j+1}-s)^{\alpha-1}ds}{2k\,\Gamma(\alpha+1)} = e_j \frac{w(\xi,\tau_1)-w(\xi,\tau_0)}{2\Gamma(\alpha+1)k^{1-\alpha}} + \frac{1}{2\Gamma(\alpha+1)} \sum_{m=0}^{j-1} e_m \frac{w(\xi,\tau_{j+1-m})-w(\xi,\tau_{j-m-1})}{k^{1-\alpha}}, \quad (5)$$

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where $e_m = (m+1)^\alpha - m^\alpha$, $m = 0,1,2,..,M$ and thus Eq. (1), can be written as

$$\int_0^{\tau_{j+1}} \frac{(\tau_{j+1}-s)^{\alpha-1} w_\tau(\xi,s)ds}{\Gamma(\alpha)} = \int_{t_0}^{\tau_j} \frac{(w(\xi,\tau_1)-w(\xi,\tau_0))(\tau_{j+1}-s)^{\alpha-1}ds}{k\Gamma(\alpha)} + \sum_{m=1}^{j} \int_{\tau_r}^{\tau_{r+1}} \frac{(w(\xi,\tau_{m+1})-w(\xi,\tau_{m-1}))(\tau_{j+1}-s)^{\alpha-1}ds}{2k\Gamma(\alpha+1)} = e_j \frac{w(\xi,\tau_1)-w(\xi,\tau_0)}{2\Gamma(\alpha+1)k^{1-\alpha}} + \frac{1}{2\Gamma(\alpha+1)} \sum_{m=0}^{j-1} e_m \frac{w(\xi,\tau_{j+1-m})-w(\xi,\tau_{j-m-1})}{k^{1-\alpha}} - w_{\xi\xi}(\xi,\tau_{j+1}) \cong f(\xi,\tau_{j+1}). \tag{6}$$

Simplifying Eq. (6), we get

$$e_0 w^{j+1}(\xi) - 2\Gamma(\alpha+1)k^{1-\alpha} w_{\xi\xi}^{j+1} = e_0 w^{j-1}(\xi) - \sum_{m=1}^{j-1} e_m \left(w^{j-m+1}(\xi) - w^{j-m-1}(\xi)\right) - 2e_j\big(w^1(\xi) - w^0(\xi)\big) + 2\Gamma(\alpha+1)k^{1-\alpha} f^{j+1}(\xi), \tag{7}$$

where $w^{j+1}(\xi) = u(\xi,\tau_{j+1})$, $f^{j+1}(\xi) = f\big(\xi,\tau_{j+1}\big)$, $e_m = (m+1)^\alpha - m^\alpha$, $m = 0,1,2,\dots,j$.

Assuming $z_0 = 2\Gamma(\alpha+1)k^{1-\alpha}$ with $e_0 = 1$, Eq. (7) is rewritten as

$$w^{j+1}(\xi) - z_0 u_{\xi\xi}^{j+1} = -e_1 w^j(\xi) + \sum_{m=1}^{j-1}(e_{m-1} - e_{m+1}) w^{j-m}(\xi) - e_j w^1(\xi) + \big(e_{j-1} + 2e_j\big) w^0(\xi) + z_0 f^{j+1}(\xi), \quad j \geq 1. \tag{8}$$

Eq. (8), is the general scheme of the problem (1)-(3), which is solved for various time levels by using the quartic B-spline method and the proposed boundary conditions. Substituting $j = 0$, in Eq. (8), u^1 is given as

$$w^1(\xi) - \frac{1}{2} z_0 w_{\xi\xi}{}^1 = w^0(\xi) + \frac{1}{2} z_0 f^1(\xi), \tag{9}$$

where $w^0 = w(\xi,0)$ is the zero time solution. The interval $[a,b]$ is partitioned into N finite elements of equal length h by the grid points ξ_i, $i = 1,2,3,\dots N$, such that $a = \xi_0 < \xi_1 < \xi_2 \dots\dots.. < \xi_N = b$ and $h = \frac{b-a}{N}$. The Quartic B-spline functions $B_i(\xi), i = -2,-1,0,\dots,N+1$ at these knots are given by:

$$B_i(\xi) = \frac{1}{h^4}\begin{cases} a_1 = (\xi-\xi_{i-2})^4, & \xi \in [\xi_{i-2},\xi_{i-1}], \\ a_2 = a_1 - 5(\xi-\xi_{i-1})^4, & \xi \in [\xi_{i-1},\xi_i], \\ a_3 = a_2 + 10(\xi-\xi_i)^4, & \xi \in [\xi_i,\xi_{i+1}], \\ (\xi_{i+3}-\xi)^4 - 5(\xi_{i+2}-\xi)^4, & \xi \in [\xi_{i+1},\xi_{i+2}], \\ (\xi_{i+3}-\xi)^4, & \xi \in [\xi_{i+2},\xi_{i+3}], \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The set of quartic B-spline functions $\{B_{-2}, B_{-1}, B_0, \dots, B_N, B_{N+1}\}$ forms a basis for the functions over the interval $[a,b]$. A global approximation $W_N(\xi,t)$ to the exact solution $w(\xi,t)$ takes the form

$$W_N(\xi,\tau) = \sum_{i=-2}^{N+1} \gamma_i(\tau) B_i(\xi), \tag{11}$$

where $\gamma_i(\tau)$ are unknown time dependent quantities to be determined from collocation, boundary and initial conditions. The nodal values W_i, the first and second derivatives $W_i{}', W_i{}''$ at the knots are obtained from Eqs. (10) and (11) in the following form,

$$\left.\begin{aligned} W_i &= W(\xi_i) = \gamma_{i-2} + 11\gamma_{i-1} + 11\gamma_i + \gamma_{i+1}, \\ W_i{}' &= W'(\xi_i) = \frac{4}{h}(-\gamma_{i-2} - 3\gamma_{i-1} + 3\gamma_i + \gamma_{i+1}), \\ W_i{}'' &= W''(\xi_i) = \frac{12}{h^2}(\gamma_{i-2} - \gamma_{i-1} - \gamma_i + \gamma_{i+1}), \end{aligned}\right\} \tag{12}$$

To develop the general scheme consider Eq. (8) and substituting the values of w^{n+1}, w_ξ^{n+1} and $w_{\xi\xi}^{n+1}$ from Eq. (12) we get

$$c_1\gamma_{i-2}^{j+1} + c_1\gamma_{i-1}^{j+1} + c_1\gamma_i^{j+1} + c_1\gamma_{i+1}^{j+1} = F_i \tag{13}$$

where $c_1 = 1 - \frac{12}{h^2}z_0$, $c_2 = 11 + \frac{12}{h^2}z_0$, $c_3 = 11 + \frac{12}{h^2}z_0$, $c_4 = 1 - \frac{12}{h^2}z_0$, $F_i = -e_1\big(\gamma_{i-2}^j + 11\gamma_{i-1}^j + \gamma_i^j + \gamma_{i+1}^j\big) + \sum_{m=1}^{j-1}(e_{m-1} - e_{m+1})\big(\gamma_{i-2}^{j-m} + 11\gamma_{i-1}^{j-m} + \gamma_i^{j-m} + \gamma_{i+1}{}^{j-m}\big) - e_j\big(\gamma_{i-2}^j + 11\gamma_{i-1}^j + \gamma_i^j + \gamma_{i+1}^j\big) + \big(e_{j-1} + 2e_j\big)(\gamma_{i-2}^0 + 11\gamma_{i-1}^0 + \gamma_i^0 + \gamma_{i+1}^0) + z_0 f^{j+1}$,
$i = 0,1,2,3,\dots,N$, $j \geq 1$. (14)

Eq. (13) is a system of $N+1$ equations in $N+4$ unknowns $\gamma_i, i = -2,-1,\cdots,N+1$. In order to get a unique solution, we eliminate the parameters $\{\gamma_{-2},\gamma_{-1},\gamma_{N+1}\}$. The values of these parameters can be obtained by collocating boundary conditions using (12). After eliminating these parameters, Eq. (13) gives a quarta diagonal system containing $N+1$ linear equations in $N+1$ unknowns. The linear system can be solved by a four-diagonal solver successively once initial time solution is obtained from Eqs. (2)-(3). Finally the approximate solution can be obtained from Eq. (11).

3. The numerical test problems

In this section we present some examples in order to test the scheme (13) for the solution of the problem defined in Eqs. (1)-(3). For comparison, all the test problems are taken from the reference (Siddiqi and Arshad, 2014).

Example 1: Consider Eqs.(1)-(3) with $\xi \in [0,1]$, and $f(\xi,\tau)$ is chosen so that the exact solution is

$$w(\xi,\tau) = (\tau+1)\sin\pi\xi. \tag{15}$$

The initial and boundary conditions are given as:
$w(\xi,0) = \sin\pi\xi, 0 \leq \xi \leq 1,$
$w(0,\tau) = 0, w_\xi(0,\tau) = \pi(\tau+1)$ and $w(1,\tau) = 0.$

Numerical simulations are performed with parameters $N = 60, k = 0.0001,\ 0.001, \alpha = \frac{1}{2}$ and different values of M. The error norms L_∞ and L_2 at various time levels are given in **(Tables 1-3)** along with the results of cubic B-spline collocation method given in (Siddiqi and Arshad, 2014). Better accuracy of the present method than cubic B-spline collocation method is evident from these tables. **(Figs. 1-2)** represent the exact and approximate solutions and error obtained through the present method corresponding to $M = 500$ and $M = 50$ respectively, whereas **(Fig. 3)** shows the quartic B-spline solutions over the time interval [0, 0.1].

Table 1: Error norms produced by the present method and Cubic B-spline collocation method (Siddiqi and Arshad, 2014) corresponding to Example-1 for $N = 60,\ \alpha = \frac{1}{2}, k = 0.0001$

	Quartic B-spline		Cubic B-spline (Siddiqi and Arshad, 2014)	
M	L_∞	L_2	L_∞	L_2
10	7.42×10^{-9}	4.72×10^{-9}	4.12×10^{-5}	3.76×10^{-6}
20	1.01×10^{-8}	6.45×10^{-9}	2.38×10^{-5}	2.17×10^{-6}
30	1.18×10^{-8}	7.55×10^{-9}	1.20×10^{-5}	1.10×10^{-6}
40	1.31×10^{-8}	8.37×10^{-9}	3.12×10^{-6}	2.85×10^{-7}
50	1.41×10^{-8}	9.02×10^{-9}	4.06×10^{-6}	3.71×10^{-7}

Table 2: Error norms produced by the present method and Cubic B-spline collocation method (Siddiqi and Arshad, 2014) corresponding to Example-1 for $N = 60, \alpha = \frac{1}{2}, k = 0.001$

	Quartic B-spline		Cubic B-spline (Siddiqi and Arshad, 2014)	
M	L_∞	L_2	L_∞	L_2
10	1.61×10^{-8}	1.02×10^{-8}	8.77×10^{-4}	8.01×10^{-5}
20	1.97×10^{-8}	1.25×10^{-8}	8.54×10^{-4}	7.79×10^{-5}
30	2.16×10^{-8}	1.38×10^{-8}	8.40×10^{-4}	7.67×10^{-5}
40	2.30×10^{-8}	1.46×10^{-8}	8.30×10^{-4}	7.58×10^{-5}
50	2.41×10^{-8}	1.53×10^{-8}	8.23×10^{-4}	7.51×10^{-5}

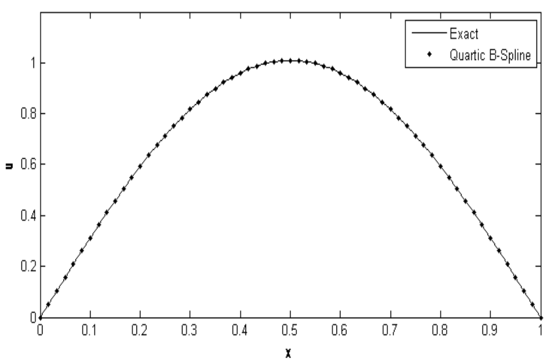

Fig.1: Plot of exact and quartic B-spline solutions for Example-1 for $h = 0.02,\ k = 0.0001$.

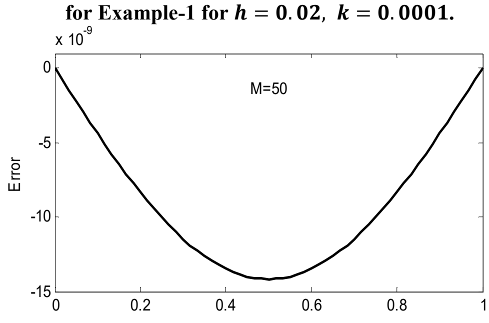

Fig.2: Error in quartic B-spline solutions

Table 3: Error norms produced by the present method and cubic B-spline collocation method (Siddiqi and Arshad, 2014) corresponding to Example-1 for $M = 10, \alpha = \frac{1}{2},\ k = 0.0001$

	Quartic B-spline		Cubic B-spline (Siddiqi and Arshad, 2014)	
N	L_∞	L_2	L_∞	L_2
10	2.10×10^{-5}	1.32×10^{-5}	2.01×10^{-3}	4.51×10^{-4}
20	1.30×10^{-6}	8.31×10^{-7}	4.28×10^{-4}	6.78×10^{-5}
30	2.58×10^{-7}	1.64×10^{-7}	1.35×10^{-4}	1.74×10^{-5}
40	8.17×10^{-8}	5.20×10^{-8}	3.21×10^{-5}	3.59×10^{-6}
50	3.34×10^{-8}	2.13×10^{-8}	1.53×10^{-5}	1.53×10^{-6}

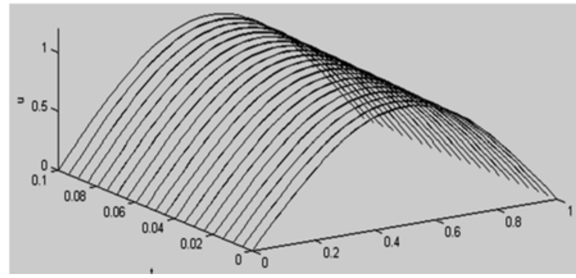

Fig.3: Quartic B-spline solutions over time interval [0, 0.1] corresponding Example-1 for $h = 0.02, k = 0.0001$

Example 2: Consider Eqs. (1)-(3) and choose $f(x,t)$ such that the exact solution is

$$w(\xi,\tau) = (\tau+1)^2 \sin \pi\xi.$$

The initial and boundary conditions are given as:

$$w(\xi,0) = \sin \pi\xi, -1 \le \xi \le 1,$$
$w(0,\tau) = 0, w_\xi(0,\tau) = -\pi(\tau+1)^2$ and
$w_\xi(1,\tau) = -\pi(\tau+1)^2$.

Numerical simulations are performed with the parameters $k = 0.001,\ 0.00125,\ \alpha = \frac{1}{2}$ and the results for different values of M. The error norms L_∞ and L_2 at various time levels are recorded in **(Tables 4-5)** along with the results of cubic B-spline collocation method (Siddiqi and Arshad, 2014). Better accuracy of the present method than cubic B-spline collocation method is evident from Tables 4-5. (**Fig. 4)** presents exact and approximate solutions obtained through the present method corresponding to $M = 500$ whereas **(Fig. 5)** shows the error plot at $M = 500$.

Table 4: Error norms produced by the present method and cubic B-spline collocation method (Siddiqi and Arshad, 2014) corresponding to Example-2 for $N = 40, \alpha = \frac{1}{2}, k = 0.001$

	Quartic B-spline		Cubic B-spline (Siddiqi and Arshad, 2014)	
M	L_∞	L_2	L_∞	L_2
10	7.29×10^{-6}	8.15×10^{-7}	5.99×10^{-4}	6.25×10^{-5}
20	1.16×10^{-5}	1.29×10^{-6}	4.33×10^{-4}	4.66×10^{-5}
30	1.50×10^{-5}	1.68×10^{-6}	7.06×10^{-4}	3.32×10^{-5}
40	1.79×10^{-5}	2.01×10^{-6}	1.31×10^{-3}	8.93×10^{-5}
50	2.06×10^{-5}	2.30×10^{-6}	2.00×10^{-3}	1.67×10^{-4}

Table 5: Error norms produced by the present method and cubic B-spline (Siddiqi and Arshad, 2014) corresponding to Example-2 for $N = 40,\ \alpha = \frac{1}{2}, k = 0.00125$

	Quartic B-spline		Cubic B-spline (Siddiqi and Arshad, 2014)	
M	L_∞	L_2	L_∞	L_2
10	1.02×10^{-5}	1.14×10^{-6}	1.00×10^{-3}	1.05×10^{-4}
20	1.63×10^{-5}	1.82×10^{-6}	7.23×10^{-4}	7.71×10^{-5}
30	2.11×10^{-5}	2.36×10^{-6}	1.11×10^{-4}	5.78×10^{-5}
40	2.53×10^{-5}	2.83×10^{-6}	1.93×10^{-3}	1.27×10^{-4}
50	2.90×10^{-5}	3.24×10^{-6}	2.87×10^{-3}	2.34×10^{-4}

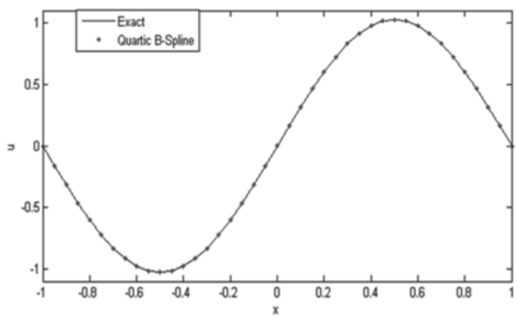

Fig. 4: Graph of exact and quartic B-spline solution corresponding to Example-2 for $h = 0.02, k = 0.0001$

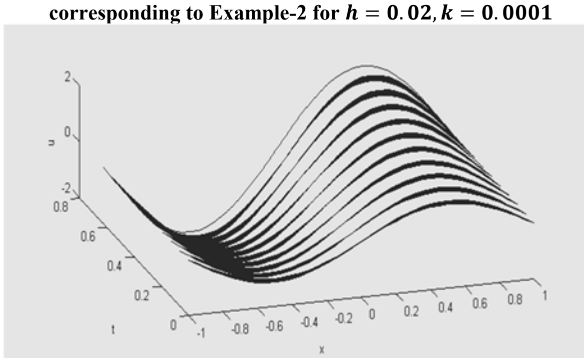

Fig. 5: Graph of quartic B-spline solutions over the time interval [0, 0.8] corresponding to Example-2

4. CONCLUSION

Quartic B-spline collocation method is used to obtain the approximate solution of parabolic type partial integro-differential equations with a weakly singular kernel. The proposed method is implemented with two test problems from literature for its validity. The accuracy of the method is examined through two error norms L_∞, L_2 and by comparison with cubic B-spline collocation method. It has been observed that the errors are sufficiently small. Simple applicability and better accuracy of the quartic B-Spline collocation method shows that this method can be employed for numerical approximation of such types of partial integro-differential equations.

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