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Modified Method for Parabolic Equations in one Dimensional with Nonlocal time Weighting Initial Condition

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Abstract: In this paper, we develop a modified version of explicit scheme based on finite difference method for the one-dimensional parabolic partial differential equations with nonlocal time weighting initial conditions. The dominancy of the Saulyev's schemes based on finite difference, over the previous explicit FTCS, Duke-Frankel, as well as implicit BTCS, Crandal's technique and Crank Nicholson's scheme has already been established, which proved to be unconditionally stable, use less CPU time and computational effort. However, in this paper a modification of Saulyev's first kind formula is developed. Main focus was to reduce error of the Saulyev's formula using proposed method. The comparison has been carried out between both methods to observe errors in different conditions and step sizes. The new modified scheme is proved to be satisfactory and unconditionally stable.

Keywords: Nonlocal Initial Condition; Nonlocal Time Weighting Initial Condition; Explicit Schemes.

1. INTRODUCTION

The partial differential equations arise in many phenomena of science and engineering. Most of these being second order partial differential equations. Two classes of second order parabolic partial differential equations had been under consideration by the authors in past, one with nonlocal initial conditions, while others consisting of nonlocal boundary conditions. This paper focusses the first type of parabolic partial differential equation with nonlocal initial conditions. Partial differential equations involving nonlocal initial conditions arise in the study of inverse heat conduction problems for determining the unknown physical parameters (Cannon and Yin, 1990, 1991, 1988) (Chadam, and Yin, 1990) (Karimi, and Vafapisheh, 1993) (Bysezewski, 1991,1992,1994) and study of diffusion in atomic reactors (Dehghan, 2004). discussed schemes for the solution of one dimensional parabolic partial differential equations, based on forward time and centered scheme explicit formula, FTCS is conditionally stable, backward time and centered space implicit formula, BTCS is unconditionally von Neumann stable Moreover, in same work, Dehghan developed two new explicit schemes using Saulyev's first kind and Saulyev's second kind formula and compared with former schemes, these two new schemes Saulyev's first and saulyev's second kind are unconditionally stable (Dehghan, 2004). In (Dehghan, 2005), Dufort–Frankel (1,3,1) was also used by Dehghan to find solution of equations (1-4), this Dufort-Frankel scheme is unconditionally stable, this scheme is modification of Richardson scheme with average time at n-1 and n+1 level was also used. (Dehghan, 2005) Richardson scheme uses central difference of both time and space which was found unconditionally unstable, a Finally, Dehghan gave two more, a three level forth order and three level sixth order explicit schemes for approximation of this problem, these two schemes are von Neumann conditionally stable (Dehghan, 2005). In (Beibalaev, 2015), authors approximated the solution of a fractional heat diffusion-wave equation without initial conditions by finite difference method, using central difference for space derivative and forward difference for time. In (Rahaman, 2015) used finite difference approximation for solution of one-dimensional diffusion equation by using forward time center space method. Crandall's implicit scheme, this Crandall's scheme is unconditionally von Neumann stable (Crandall,1955) and Crank-Nicholson implicit technique using strong maximum principle with weighting averages (Crank, and Nicolson, 1974), this Crank-Nicholson scheme is unconditionally von Neumann stable (Crank, and Nicolson, 1974).

2. MATERIAL AND METHODS

2.1. General description:

In this paper, we consider the problem of finding unknown $u(x,t)$ in the parabolic partial differential equation.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\emptyset(x,t), \quad 0<x<1, 0<t\leq T, \quad (1)$$

With Dirichlet boundary conditions.

$$u(0,t)=g_0(t), \quad 0<t\leq 1, \quad (2)$$

$$u(1,t)=g_1(t), \quad 0<t\leq 1, \quad (3)$$

Nonlocal time weighting initial conditions.

$$u(x,0)=\sum_{j=1}^{N}\beta_j(t)u(x,T_j)+\varphi(x), \quad 0<x<1, \quad 0<T_1,T_2,\dots,T_N=T, \quad (4)$$

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2.2. Existing schemes:

Various explicit and implicit schemes are present to solve the model problem (1-4). The FTCS, Duke-Frankel, Saulyev's First Kind Formula, Saulyev's Second Kind Formula, (1,3,1) Fourth Order and (1,3,3) Sixth Order, are among explicit schemes. We present the Saulyev's First kind formula and its new modification in the section.

2.2. Saulyev's First Kind Formula:

The direct substitutions of finite difference approximation for derivatives in (1) for subjected schemes resulted the following.

$$u_i^{n+1} = \frac{1}{1+s}[su_{i-1}^{n+1} + (1-s)u_i^n + su_{i+1}^n + k\emptyset_i^n] \qquad (5)$$

This scheme is unconditionally stable for all s > 0, .

2.3. Proposed scheme:

In this scheme, we use forward difference quotient for time derivative approximation and average of Saulyev's First kind formula and its modification for space derivative approximation given as;

$$u_t = \frac{u_i^{n+1} - u_i^n}{k} \qquad (6)$$

$$u_{xx} = \frac{1}{2}\frac{u_{i-1}^{n+1} - u_i^n - u_i^{n+1} + u_{i+1}^n}{h^2} + \frac{1}{2}\frac{u_{i-1}^{n-1} - u_i^n - u_i^{n-1} + u_{i+1}^n}{h^2} \qquad (7)$$

Now, substituting (6) and (7) in (1) yields the following;

$$u_i^{n+1} = \frac{1}{(2+s)}[su_{i-1}^{n+1} + (2-2s)u_i^n + 2su_{i+1}^n - su_i^{n-1} + su_{i-1}^{n-1} + 2k\emptyset_i^n] \quad (8)$$

3. COMPARISON AND NUMERICAL EXAMPLES:

To examine our new modified scheme with Saulyev's first kind formula, we perform a numerical test in this section with following example.

$\emptyset(x,t) = (-1+\pi^2)\sin(\pi x)\exp(-t)$,
$u(x,0) = \beta_1 u(x,T_1) - \beta_2 u(x,T_2) + \varphi$, is Initial Condition,

$\varphi(x) = \sin(\pi x)\,(1 - e^{T_1} + e^{T_2})$,
$g_0(t) = 0, g_1(t) = 0$, are boundary condition, $0 < T_1 < T_2 = T = 1$.

Note that: L_2 error norm is defined as $L_2 = \|u - \tilde{u}\|_2 = \sqrt{\frac{1}{m}\sum_{i=1}^{m}|u_i - \tilde{u}_i|^2}$ (9)in (9), u is exact solution and $\tilde{u}$ is approximate solution

4. RESULTS AND DISCUSSION

In this section, we have summarized the results regarding the values of s=1/4, 1/3 and 1/2, and regarding time $T_1 = 0.5$ *and* $T_2 = 1.0$, $T_1 = 0.5$ *and* $T_2 = 0.75$. The L_2 error norm in (9) are obtained for the numerical example at T=1 with different values of h.

4.1. Tables Regarding $s = \frac{k}{h^2}$

Table: 01

L_2error norm at T=1, with$s = \frac{1}{4}, \beta_1 = 1$ *and* $\beta_2 = -1$, *while* $T_1 = 0.5$ *and* $T_2 = 1.0$ are given in this table.

Number of time steps n	Step size 'h'	L_2 Norm Saulyev's First Kind	L_2 Norm New Modified	Maximum Saulyev's First kind	Maximum New Modified
400	0.1	8.6953e-003	7.4326e-003	3.9763e-003	3.3240e-003
1600	0.05	3.0981e-003	2.5289e-003	1.0566e-003	7.9972e-004
6400	0.025	1.1374e-003	7.4375e-004	3.0099e-004	1.6631e-004
25600	0.0125	5.1269e-004	3.4478e-005	8.7516e-005	5.4514e-006

Table 02

L_2error norm at T=1, with$s = \frac{1}{3}, \beta_1 = 1$ *and* $\beta_2 = -1$, *while* $T_1 = 0.5$ *and* $T_2 = 1.0$ are given in this table.

Number of time steps n	Step size 'h'	L_2 Norm Saulyev's First Kind	L_2 Norm New Modified	Maximum Saulyev's First kind	Maximum New Modified
300	0.1	9.1175e-003	7.3935e-003	4.2297e-003	3.3065e-003
1200	0.05	3.3325e-003	2.5162e-003	1.1673e-003	7.9569e-004
4800	0.025	1.3296e-003	7.3939e-004	3.5714e-004	1.6533e-004
19200	0.0125	6.8156e-004	3.2957e-005	1.1533e-004	5.2110e-006

Table 03

L_2error norm at T=1, with$s = \frac{1}{2}, \beta_1 = 1$ and $\beta_2 = -1$, $while$ $T_1 = 0.5$ and $T_2 = 1.0$ are given in table 03.

Number of time steps n	Step size 'h'	L_2 Norm Saulyev's First Kind	L_2 Norm New Modified	Maximum Saulyev's First kind	Maximum New Modified
200	0.1	1.0005e-002	7.3143e-003	4.7336e-003	3.2711e-003
800	0.05	3.8568e-003	2.4906e-003	1.4045e-003	7.8758e-004
3200	0.025	1.7511e-003	7.3065e-004	4.7112e-004	1.6338e-004
12800	0.0125	1.0196e-003	2.9916e-005	1.7095e-004	4.7301e-006

4.2. Tables Regarding T_1 and T_2

Table 04

L^2error norm at T=1, with$s = 0.25, \beta_1 = 1$ and $\beta_2 = -1$, $while$ $T_1 = 0.25$ and $T_2 = 1.0$ are given in this table.

Number of time steps n	Step size 'h'	L_2 Norm Saulyev's First Kind	L_2 Norm New Modified	Maximum Saulyev's First kind	Maximum New Modified
400	0.1	8.7129e-003	7.4487e-003	3.9840e-003	3.3311e-003
1600	0.05	3.1096e-003	2.5407e-003	1.0600e-003	8.0345e-004
6400	0.025	1.1440e-003	7.5216e-004	3.0253e-004	1.6819e-004
25600	0.0125	5.1372e-004	4.0438e-005	8.8160e-005	6.3938e-006

Table 05

L_2error norm at T=1, with$s = 0.25$, $\beta_1 = 1$ and $\beta_2 = -1$, $while$ $T_1 = 0.5$ and $T_2 = 0.75$ are given in this table.

Number of time steps n	Step size 'h'	L_2 Norm Saulyev's First Kind	L_2 Norm New Modified	Maximum Saulyev's First kind	Maximum New Modified
400	0.1	8.6661e-003	7.4061e-003	3.9636e-003	3.3121e-003
1600	0.05	3.0792e-003	2.5095e-003	1.0509e-003	7.9358e-004
6400	0.025	1.1266e-003	7.2989e-004	2.9845e-004	1.6321e-004
25600	0.0125	5.1115e-004	2.4650e-005	8.6497e-005	3.8976e-006

4.3. Table Regarding Computational Times

Table: 06

Using Processor:Intel(R) Core (TM) M-5Y10c CPU @ 0.80GHz 1.00 GHz, Installed memory (RAM): 4.00 GB.

Number of time steps n	Step size 'h'	Computation Time of Saulyev's First Kind	Computation Time of New Modified
400	0.1	0.025 seconds	0.037 seconds
1600	0.05	0.094 seconds	0.134 seconds
6400	0.025	0.763 seconds	0.949 seconds
25600	0.0125	6.052 seconds	7.536 seconds

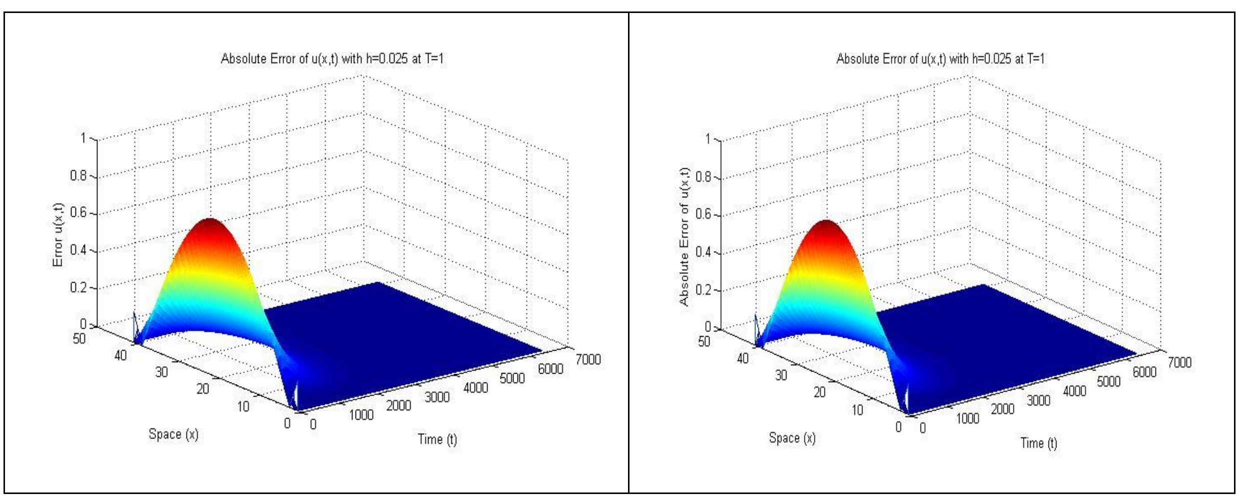

Fig. 1-a

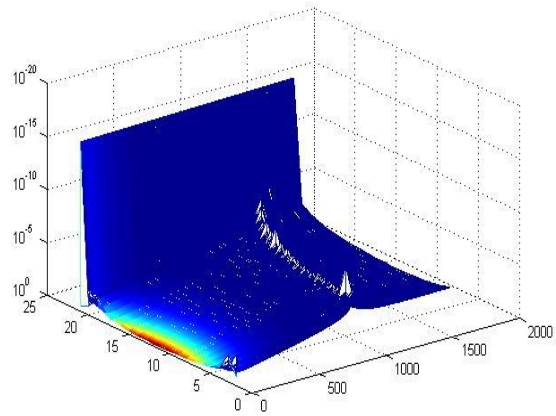

Fig. 2-a

Fig. 1-b

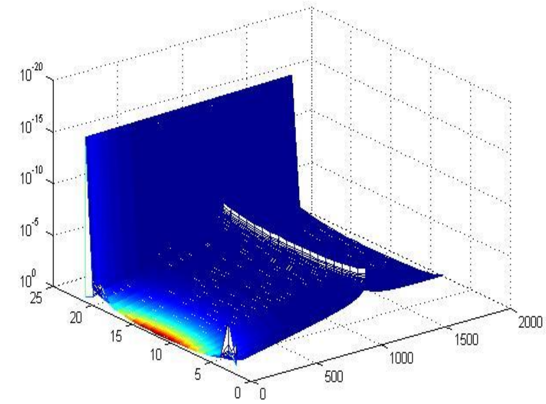

Fig.2-b

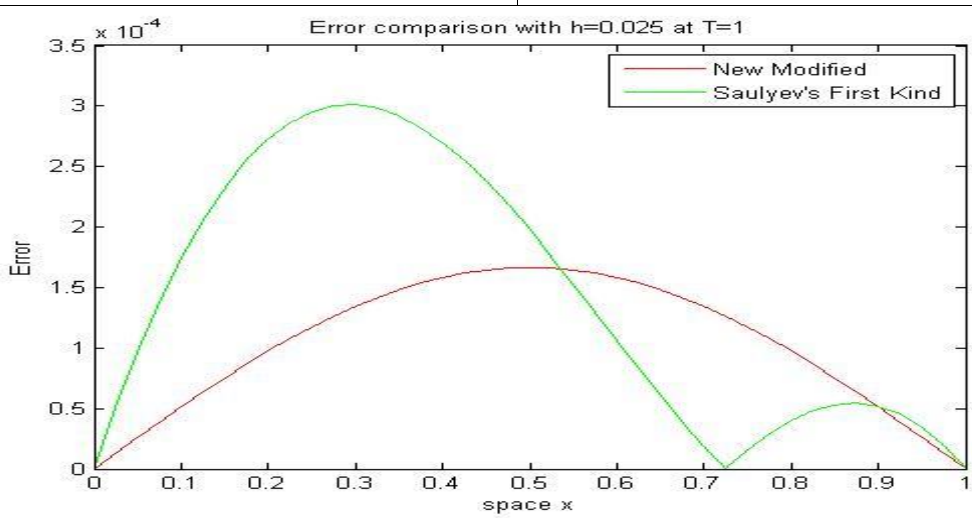

Fig. 3

The results for different values of s are shown in Tables 1-3. For s=1/4 in **(Table 1),** it can be seen that theL_2 error norm at T=1 with h=0.05 for the new modified method is less than that of Saulyev's First kind formula. With h = 0.025 in Table 1 under same conditions, the L_2 norm for new modified method is 7.4375e-004, which is also less than the corresponding norm of the Saulyev's First kind formula.

Similarly, for s =1/5 and s=1/3, respectively, it is evident from **(Tables 2 and 3)** that the error norms for the new modified method are smaller than those by the Saulyev's First kind formula. The errorfurther becomes smaller as values of h decrease.For a fixed T=1, the maximum error norms across all x values are also described in Tables 1-3. It is clear that the maximum error norms obtained at each step size h by New Modified step size are also smaller than Saulyev's first kind. For a fixed value of s=1/4, results with different values of time $T_1$and T_2 in nonlocal initial condition (17) such as $T_1 = 0.25, T_2 = 1.0$ and $T_1 = 0.5, T_2 = 0.75$ are shown in Table 4-5 respectively, it can be seen that the L_2 error norm at T=1 with h=0.05 for the new modified method is less than that of Saulyev's First kind formula, however the maximum error at the same tables also provide the sufficient evidence for the New modified method is better approximation for the unknown u(x, t).

(Table 4-6) compares computational times by Saulyev's First kind and New modified methods. It can be seen that there is little difference in CPU times by Saulyev's First kind and New modified methods. Whereas the accuracy of new method is better than the Saulyev's method **(Tables 1-5).**

(Fig 1a and b) show the over-all errors by Saulyev's First kind and New modified methods. There is a little difference in the peak errors across t = 1000-1500. This can be seen from Figures 2 (a and b), where the z-axis is reversed. To observe the error difference, the absolute errors of the final iteration, i.e. at T =1 are shown in **(Fig. 1-2 and Fig. 3),** in particular, show the improved performance of the new method.

5. CONCLUSION

In overall comparison of Saulyev's First kind and New modified methods regarding s, T_1, T_2, β_1 *and* β_2, the performance of New method is quite satisfactory leading to more accuracy. The errors obtained are smaller than at each step size h. The accuracy also depends of values of s, T_1, T_2, β_1 *and* β_2. It has been observed that the errors in the new method are always smaller than the errors of Saulye's First kind formula, moreover the L_2 error norm of new method also smaller than Saulyev's first kind formula in all conditions. So the new method can be used in place of saulyev's first kind formula to achieve better accuracy.

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