Sindh Univ. Res. Jour. (Sci. Ser.) Vol. 48 (2) 383-388 (2016)

SINDH UNIVERSITY RESEARCH JOURNAL (SCIENCE SERIES)

New generalisation of interval-valued fuzzy filters of ordered semigroups

H.U. KHAN[++], F. M. KHAN*, A. KHAN**, N. H. SARMIN***

Department of Mathematics, University of Malakand, at Chakdara, Pakistan.

Received 16th March 2015 and Revised 23rd February 2016

Abstract: **Abstract.** It is not always possible for membership functions of type $f : X \rightarrow [0,1]$ to associate with each point x in a set X a real number in the closed unit interval without the loss of some useful information. Keeping this point mind, interval-valued fuzzy set has been introduced by extended the grade of membership to an interval value $\widetilde{t} = [t^-, t^+] \in D[0,1]$ rather than a single point and introduced a theory of fuzzy sets. This is one of the most important and useful generalisation of Zadeh's fuzzy set theory and is called interval-valued (in short i-v) fuzzy set theory. Thereafter, the notion of i-v fuzzy set has been hired in different fields of algebra, especially in semigroups and ordered semigroups (for short o.s). We consider ordered semigroup and introduce another generalisation of fuzzy filters. Moreover, this new type i-v fuzzy left (right) filters are linked with i-v fuzzy left (right) filters. Further, ordinary filters are linked with this new notion by using level subsets.

Keywords. Filters; Ordered semigroup; fuzzy filters; i.v fuzzy set; i.v fuzzy filter.

1. INTRODUCTION

Zadeh (Zadeh, 1975, Grattan-Guinness, 1976 and John, 1975) independently coined the idea of a fuzzy set with an i-v membership function. Iinterval-valued fuzzy set theory is a natural extension of Zadeh's fuzzy set theory (Zadeh, 1965) where the value of the membership function is a closed sub interval in [0,1] rather than a single point. In addition, by using fuzzy set theory in the structure of groups, Rosenfeld gave the notion of fuzzy sub-group (Rosenfeld, 1971). Later on (Biswas, 1994) applied the concept of i-v fuzzy set in algebra and presented a generalisation fuzzy sub-group which is called i-v fuzzy sub-group. Further, (Davvaz, 1999) extended the idea introduced by (Biswas, 1994) to hypergroups and defined i-v fuzzy subhypergroup. In addition, in semigroups, Narayanan and Manikantan (Narayanan and Manikantan, 2006), gave the notion of i-v fuzzy ideal theory. They also discussed characterisation of semigroups in terms of these notions. Moreover, (Shabir and Khan, 2008) extended this notion of (Narayanan and Manikantan, 2006), to ordered semigroups. Furthermore, (Murali, 2004) gave the idea that a fuzzy point x_t belongs to a fuzzy subset f if $f(x) \geq t$ (where $t \in (0,1]$. On the other hand according to Pao-Ming and Ying-Ming (Pao-Ming and Ying-Ming, 1980), if $f(x) + t > 1$, then a we say that there is a quasi-coincidence relation between x_t and f. This concept of (Pao-Ming and Ying-Ming, 1980) played a key role in the comprehensive study of fuzzy algebra. In addition, (Khan *et al.*, 2013) generalised the concept of i-v fuzzy bi-ideals of ordered semigroup in terms of i-v fuzzy point. On the other hand, (Kehayopulu and Tsingelis, 2002) initiated the idea of fuzzy filters in ordered semigroups. In addition, (Davvaz *et al.*, 2013) introduced i-v fuzzy left (right) filters of type $(\in, \in \vee q_{\widetilde{k}})$ in ordered semigroup. Further, in BL-algebras (Yin and Zhan, 2010) defined $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filters and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy filters and provided some interesting results in terms of these new concepts. Moreover, (Ma *et al.*, 2011) discussed $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of BCI-algebras and proved several interesting results of BCI-algebras. In addition, (Khan *et al.*, 2011) applied the concepts of (Yin and Zhan, 2010) in ordered semigroups and studied fuzzy left ideals and fuzzy interior ideals of type $(\in_\gamma, \in_\gamma \vee q_\delta)$. Further, in the structure of ordered semigroup (Khan *et al.*, 2014) discussed in detail the notion of fuzzy generalized bi-ideals of type $(\in_\gamma, \in_\gamma \vee q_\delta)$.

In the present study, we considered the concept given in (Yin and Zhan, 2010) in ordered semigroup to provide another generalisation of the concept given in (Davvaz *et al.*, 2013). This new generalisation is called i-v fuzzy filters of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$. Further, this new

[++]Correspondence Author: Email: hidayatullak@yahoo.com
*Department of Mathematics and Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan.
**Department of Mathematics, Abdul Wali Khan University Mardan, Mardan, Khyber Pakhtunkhwa, Pakistan,
***Department of Mathematical Sciences, Faculty of Science, University Teknologi Malaysia 81310 UTM Johor Bahru, Johor, Malaysia.

concept is linked ordinary i-v fuzzy left filters by imposing a condition on an i-v fuzzy subset which is under consideration. Further, we have also shown that the intersection of finite number of i-v $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy filters is again i-v $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy filters. In addition, the most important part of this research is to link ordinary fuzzy left (right) filters with these newly defined concept by using the level subsets.

2. PRELIMINARIES

The fundamental results concepts that are essential for this paper are provided in this section. Throughout this paper ordered semigroup is represented by S unless otherwise stated and $x \cdot y$ by xy for each elements x, y in S.

Let $\leq$ is an ordered relation on the elements of S, where S represents a semigroup. If $(S, \leq)$ is a partially ordered set (poset) and $b \geq a$ implies $b \cdot x \geq a \cdot x$ and $x \cdot b \geq x \cdot a$ for all $x, a, b \in S$, then the triple $(S, \cdot, \leq)$ is called ordered semigroup.

An element $x \cdot y$ of S will be represented by xy throughout in this paper.

The set $(A] := \{t \in S \mid t \leq h \text{ for some } h \in A\}$ is defined for a subset A of S. And $AB := \{ab \mid a \in A, b \in B\}$ for a non-empty subsets A and B of S.

If $\phi \neq A \subseteq S$ and $A^2 \subseteq A$, then is known as a sub-semi-group.

A sub-semi-group F is called a left filter if for all $a, b \in F$ the following conditions hold:

(b_1) $a \leq b \rightarrow b \in F$,

(b_2) $ab \in F \Rightarrow a \in F$.

And F is called right filter if (b_1) and the following condition hold:

(b_3) (for all $a, b \in F$) $ab \in F \Rightarrow b \in F$.

If (b_1), (b_2) and (b_3) hold for all $a, b \in F$, then is called a filter of S.

Next, we give some important fuzzy concepts.

Let μ be a fuzzy subset. Then then μ is called a fuzzy sub-semi-group of S, if for all $x, y \in S$ we have: $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$.

If μ is given to be a fuzzy sub-semi-group, then μ is called a fuzzy left filter if the following conditions hold for all $x, y \in S$:

(b_4) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$,

(b_5) $\mu(x) \geq \mu(xy)$.

If (b_4) holds and $\mu(y) \geq \mu(xy)$ for all $x, y \in S$, then μ is called a fuzzy right filter of S.

And μ is called a fuzzy filter of S, if, it is both fuzzy left filter and fuzzy right filter.

In the following lines some i-v fuzzy set's concepts are reminded.

If $0 \leq a^- \leq a^+ \leq 1$, then the interval $[a^-, a^+]$ is called *interval number* and is denoted by $\tilde{a}$ and $D[0,1]$ is the set of all interval numbers. Let $i \in I$ and $\tilde{l}_k = [l_k^-, l_k^+]$, $\tilde{m}_k = [m_k^-, m_k^+] \in D[0,1]$. Then:

$\text{r}\max\{\tilde{l}_k, \tilde{m}_k\} = [\max(l_i^-, m_i^-), \max(l_i^+, m_i^+)]$,

$\text{r}\min\{\tilde{l}_k, \tilde{m}_k\} = [\min(l_i^-, m_i^-), \min(l_i^+, m_i^+)]$.

An i-v fuzzy subset is defined and represented on X as $A = \{(x, \tilde{F}_A(x) = [F_A^-(x), F_A^+(x)]) : x \in X\}$, where $0 \leq F_A^-(x) \leq F_A^+(x) \leq 1$ and $F_A^-(x)$, $F_A^+(x)$ are representing fuzzy membership's grade of an element x of X. The function $\tilde{F}_A : X \rightarrow D[0,1]$ represents membership's grade of an element x in X to A, where $D[0,1]$ denotes the family of all closed subintervals of the unit closed interval $[0,1]$.

If $a \in [0,1]$, then a can be represented as an interval $[a, a]$ and we call it point interval. Consequently, we observe that i-v fuzzy set theory is one of the natural generalization of the theory of fuzzy sets.

For an i-v fuzzy set A we define $\tilde{F}_A(x) + \tilde{t} = [F_A^-(x) + t^-, F_A^+(x) + t^+]$ $\forall$ $x \in X$, where $\tilde{t} = [t^-, t^+] \in D(0,1]$. In particular, we write $\tilde{F}_A(x) + \tilde{t} > \tilde{1}$ whenever, $F_A^-(x) + t^- > 1$ and $F_A^+(x) + t^+ > 1$.

If A is i-v fuzzy subset of S and

$$\tilde{F}_A(y) := \begin{cases} \tilde{t}, & \text{if } y \leq x, \\ \tilde{0} = [0,0], & \text{otherwise} \end{cases}$$

then A is called i-v fuzzy point and is denoted by $x_{\tilde{t}}$, where x is the support and $\tilde{t}$ is the value of the i-v fuzzy point $x_{\tilde{t}}$.

An i-v fuzzy point $x_{\tilde{t}}$ is said to belongs to an i-v fuzzy subset A, if $\tilde{t} \leq \tilde{F}_A(x)$ and we write $x_{\tilde{t}} \in A$. On the other hand, if $\tilde{1} < \tilde{F}_A(x) + \tilde{t}$, then $x_{\tilde{t}}$ is quasi-coincident with A and is denoted by $x_{\tilde{t}} qA$. Whenever, $x_{\tilde{t}}$ belongs to A or $x_{\tilde{t}}$ is coincident with A, then we represent this relation as $x_{\tilde{t}} \in \vee qA$.

The set of the form $U(A; \tilde{t}) = \{x \in X \mid \tilde{F}_A(x) \geq \tilde{t}\}$, where $\tilde{0} < \tilde{t} \leq \tilde{1}$ is called the *level set* of an i-v fuzzy subset A.

If the following conditions hold for all $x, y \in S$:

(b_6) $x \le y \to \widetilde{F}_A(x) \le \widetilde{F}_A(y)$,

(b_7) $\text{r}\min\{\widetilde{F}_A(x), \widetilde{F}_A(y)\} \le \widetilde{F}_A(xy)$,

(b_8) $\widetilde{F}_A(xy) \le \widetilde{F}_A(x)$,

then A is said to be i-v fuzzy left filter.

On the other hand A is known as i-v fuzzy right filter if conditions (b_6), (b_7) and the following condition hold:

(b_9) $\widetilde{F}_A(xy) \le \widetilde{F}_A(y)$ for all $x, y \in A$.

A will be known as i-v fuzzy filter if the conditions (b_6), (b_7), (b_8) and (b_9) hold simultaneously for all $x, y \in A$.

3. GENERALISED INTERVAL-VALUED FUZZY FILTERS

A new generalized form of the ordinary i-v fuzzy filters is given here in this part of the paper. We have contributed another useful generalisation of i-v fuzzy left (resp. right) filters and have provided some characterization results of ordered semigroup in terms of this new concept.

Throughout this paper, we take $\widetilde{\gamma}$ and $\widetilde{\delta}$ from $D[0,1]$ and $\widetilde{\gamma} < \widetilde{\delta}$.

If $\widetilde{F}_A(x) \ge \widetilde{t} > \widetilde{\gamma}$, then we say that the i-v fuzzy point $x_{\widetilde{t}}$ $\widetilde{\gamma}$-belongs to A and is denoted by $x_{\widetilde{t}} \in_{\widetilde{\gamma}} A$. On the other hand, if $\widetilde{F}_A(x) + \widetilde{t} > 2\widetilde{\delta}$, then we say that $x_{\widetilde{t}}$ has $\widetilde{\delta}$-quasi-coincidence relation with A and we write $x_{\widetilde{t}} q_{\widetilde{\delta}} A$. If $x_{\widetilde{t}} \in_{\widetilde{\gamma}} A$ or $x_{\widetilde{t}} q_{\widetilde{\delta}} A$, then this is represented as $x_{\widetilde{t}} \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}} A$. If $\alpha \in \{\in_{\widetilde{\gamma}}, q_{\widetilde{\delta}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}}, \in_{\widetilde{\gamma}} \wedge q_{\widetilde{\delta}}\}$ and the relation $x_{\widetilde{t}} \alpha A$ does not satisfy, then we write $x_{\widetilde{t}} \overline{\alpha} A$.

3.1 Definition

If A satisfy all the below given assertions simultaneously for all $x, y \in S$ and $\widetilde{t}, \widetilde{t}_1, \widetilde{t}_2 \in D(\widetilde{\gamma}, 1]$:

(c_1) If $x \le y$ and $x_{\widetilde{t}} \in_{\widetilde{\gamma}} A$, then $y_{\widetilde{t}} \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}} A$,

(c_2) If $x_{\widetilde{t}_1} \in_{\widetilde{\gamma}} A$ and $y_{\widetilde{t}_2} \in_{\widetilde{\gamma}} A$, then $(xy)_{\text{r}\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}} A$,

(c_3) If $(xy)_{\widetilde{t}} \in_{\widetilde{\gamma}} A$, then $x_{\widetilde{t}} \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}} A$.

Then A is called i-v fuzzy left filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$, whereas, if (c_1), (c_2) and the below stated condition hold simultaneously, then A will be called i-v fuzzy right filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$:

(c_4) If $(xy)_{\widetilde{t}} \in_{\widetilde{\gamma}} A$, then $y_{\widetilde{t}} \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}} A$.

If all the assertions (c_1), (c_2), (c_3) and (c_4) of Definition 3.1 hold simultaneously, then we call A an i-v fuzzy filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$. In other words A will an i-v fuzzy filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$ if A is both i-v fuzzy left and right filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$.

If we take $\widetilde{\gamma} = [0,0]$ and $\widetilde{\delta} = [0.5, 0.5]$, then the $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$-fuzzy left (resp. right) filter of S is called an $(\in, \in \vee q)$-fuzzy left (resp. right) filter of S.

3.2 Example

Let $S = \{a, b, c, d, e, f\}$ with the order relation $(\le)$ given below and the multiplication given in Table 1:

$$\le := \left\{ \begin{matrix} (a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,d), \\ (a,e),(d,e),(b,f),(c,f),(c,e),(f,e) \end{matrix} \right\}.$$

Table : 1

$\cdot$	a	b	c	d	e	f
a	a	b	b	d	e	f
b	b	b	b	b	b	b
c	b	b	b	b	b	b
d	d	b	b	d	e	f
e	e	f	f	e	e	f
f	f	f	f	f	f	f

Then $(S, \cdot, \le)$ is an ordered semigroup. Define an i-v fuzzy subset A on S by

$$\widetilde{F}_A(x) = \begin{cases} [0.50, 0.60], & \text{if } x = a, \\ [0.70, 0.80], & \text{if } x \in \{b, c, f\}, \\ [0.80, 0.90], & \text{if } x = d, \\ [0.40, 0.50], & \text{if } x = e. \end{cases}$$

It is easy to check that A is an i-v $(\in_{[0.3,0.4]}, \in_{[0.3,0.4]} \vee q_{[0.4,0.5]})$-fuzzy filter of S.

3.3 Theorem

An i-v fuzzy subset A is an i-v fuzzy left filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$ if and only if all the assertions stated below hold simultaneously for all $x, y \in S$.

(1) $x \le y \Rightarrow \text{r}\min\{\widetilde{F}_A(x), \widetilde{\delta}\} \le \text{r}\max\{\widetilde{F}_A(y), \widetilde{\gamma}\}$,

(2) $\text{r}\min\{\widetilde{F}_A(x), \widetilde{F}_A(y), \widetilde{\delta}\} \le \text{r}\max\{\widetilde{F}_A(xy), \widetilde{\gamma}\}$,

(3) $\text{r}\min\{\widetilde{F}_A(xy), \widetilde{\delta}\} \le \text{r}\max\{\widetilde{F}_A(x), \widetilde{\gamma}\}$.

Proof. Take A is an i-v fuzzy left filter of type $(\in_{\widetilde{\gamma}}, \in_{\widetilde{\gamma}} \vee q_{\widetilde{\delta}})$ and (1) is not true for all $x, y \in S$ such that $x \le y$. Then $\text{r}\min\{\widetilde{F}_A(a), \widetilde{\delta}\} > \text{r}\max\{\widetilde{F}_A(b), \widetilde{\gamma}\}$ for some $a \le b$ ($a, b \in S$). Therefore, there exists $\widetilde{t} \in D(\widetilde{\gamma}, 1]$ such that

$\text{r}\min\{\widetilde{F}_A(a), \widetilde{\delta}\} \ge \widetilde{t} > \text{r}\max\{\widetilde{F}_A(b), \widetilde{\gamma}\}$.

It follows that $a_{\widetilde{t}} \in_{\widetilde{\gamma}} A$ but $b_{\widetilde{t}} \overline{\in}_{\widetilde{\gamma}} A$ and $\overline{q}_{\widetilde{\delta}} A$, a contradiction and hence for all $x \le y$ ($x, y \in S$) we have $\text{r}\min\{\widetilde{F}_A(x), \widetilde{\delta}\} \le \text{r}\max\{\widetilde{F}_A(y), \widetilde{\gamma}\}$.

Let us consider on contrary that (2) does not hold, i.e., $\text{r min}\{\tilde{F}_A(a),\tilde{F}_A(b),\tilde{\delta}\} > \text{r max}\{\tilde{F}_A(ab),\tilde{\gamma}\}$ for some $a,b \in S$. Then there exist some $\tilde{s} \in D(\tilde{\gamma},1]$ such that $\text{r min}\{\tilde{F}_A(a),\tilde{F}_A(b),\tilde{\delta}\} \geq \tilde{s} > \text{r max}\{\tilde{F}_A(ab),\tilde{\gamma}\}$ and hence $a_{\tilde{s}} \in_{\tilde{\gamma}} A$, $b_{\tilde{s}} \in_{\tilde{\gamma}} A$ but $(ab)_{\tilde{s}} \overline{\in}_{\tilde{\gamma}} A$ and $(ab)_{\tilde{s}} \overline{q}_{\tilde{\delta}} A$, a contradiction and thus (2) is valid for all $x, y \in S$.

To show that Condition (3) is valid we suppose on contrary that there exist $a,b \in S$ such that

$\text{r min}\{\tilde{F}_A(ab),\tilde{\delta}\} > \text{r max}\{\tilde{F}_A(a),\tilde{\gamma}\}$.

Then

$\text{r min}\{\tilde{F}_A(ab),\tilde{\delta}\} \geq \tilde{t} > \text{r max}\{\tilde{F}_A(a),\tilde{\gamma}\}$

for some $\tilde{t} \in D(\tilde{\gamma},1]$ and it follows that $(ab)_{\tilde{t}} \in_{\tilde{\gamma}} A$ but $a_{\tilde{t}} \overline{\in}_{\tilde{\gamma}} A$ and $a_{\tilde{t}} \overline{q}_{\tilde{\delta}} A$. Again a contradiction and hence $\text{r min}\{\tilde{F}_A(xy),\tilde{\delta}\} \leq \text{r max}\{\tilde{F}_A(x),\tilde{\gamma}\}$ for all $x, y \in S$.

Conversely, let A be an i-v fuzzy subset of S and assertions (1), (2) and (3) hold simultaneously. If $x, y \in S$, $\tilde{t} \in D(\tilde{\gamma},1]$ such that $y \geq x \in_{\tilde{\gamma}} A$, then by (1) we have

$$\begin{aligned}\text{r max}\{\tilde{F}_A(y),\tilde{\gamma}\} &\geq \text{r min}\{\tilde{F}_A(x),\tilde{\delta}\}\\ &\geq \text{r min}\{\tilde{t},\tilde{\delta}\}\\ &= \begin{cases}\tilde{t}, & \text{if } \tilde{t} \leq \tilde{\delta}\\ \tilde{\delta}, & \text{if } \tilde{t} > \tilde{\delta}.\end{cases}\end{aligned}$$

From the above inequality we see that $y_{\tilde{t}} \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}} A$ and hence (c_1) holds.

Next, if $x, y \in S$ and $\tilde{t}_1, \tilde{t}_2 \in D(\tilde{\gamma},1]$ such that $x_{\tilde{t}_1} \in_{\tilde{\gamma}} A$, $y_{\tilde{t}_2} \in_{\tilde{\gamma}} A$, then by (2) we have

$$\begin{aligned}\text{r max}\{\tilde{F}_A(xy),\tilde{\gamma}\} &\geq \text{r min}\{\tilde{F}_A(x),\tilde{F}_A(x),\tilde{\delta}\}\\ &\geq \text{r min}\{\tilde{t}_1,\tilde{t}_2,\tilde{\delta}\}\\ &= \begin{cases}\text{r min}\{\tilde{t}_1,\tilde{t}_2\}, & \text{if } \text{r min}\{\tilde{t}_1,\tilde{t}_2\} \leq \tilde{\delta},\\ \tilde{\delta}, & \text{if } \text{r min}\{\tilde{t}_1,\tilde{t}_2\} \tilde{>} \delta,\end{cases}\end{aligned}$$

showing that $(xy)_{\text{r min}\{\tilde{t}_1,\tilde{t}_2\}} \in_{\tilde{\gamma}} A$ or $(xy)_{\text{r min}\{\tilde{t}_1,\tilde{t}_2\}} q_{\tilde{\delta}} A$. Therefore, $(xy)_{\text{r min}\{\tilde{t}_1,\tilde{t}_2\}} \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}} A$ and (c_2) holds.

Let $x, y \in S$ and $\tilde{t} \in D(\tilde{\gamma},1]$ such that $(xy)_{\tilde{t}} \in_{\tilde{\gamma}} A$, then by (3) we have

$$\begin{aligned}\text{r max}\{\tilde{F}_A(x),\tilde{\gamma}\} &\geq \text{r min}\{\tilde{F}_A(xy),\tilde{\delta}\}\\ &\geq \text{r min}\{\tilde{t},\tilde{\delta}\}\\ &= \begin{cases}\tilde{t}, & \text{if } \tilde{t} \leq \tilde{\delta},\\ \tilde{\delta}, & \text{if } \tilde{t} > \tilde{\delta}.\end{cases}\end{aligned}$$

From the above inequality we observe that $x_{\tilde{t}} \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}} A$ and therefore (c_3) also holds. Hence proved as required.

3.4 Theorem

A is an i-v fuzzy right filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ of S if and only if assertions (1), (2) of Theorem 3.3 and the assertion stated below hold simultaneously for all $x, y \in S$:

(4) $\text{r max}\{\tilde{F}_A(y),\tilde{\gamma}\} \geq \text{r min}\{\tilde{F}_A(xy),\tilde{\delta}\}$.

Proof. Let $\tilde{t} \in D(\tilde{\gamma},1]$ and $(xy)_{\tilde{t}} \in_{\tilde{\gamma}} A$ for $x, y \in S$. Then by (4) we have

$$\begin{aligned}\text{r max}\{\tilde{F}_A(y),\tilde{\gamma}\} &\geq \text{r min}\{\tilde{F}_A(xy),\tilde{\delta}\}\\ &\geq \text{r min}\{\tilde{t},\tilde{\delta}\}\\ &= \begin{cases}\tilde{t}, & \text{if } \tilde{t} \leq \tilde{\delta},\\ \tilde{\delta}, & \text{if } \tilde{t} > \tilde{\delta}.\end{cases}\end{aligned}$$

The above inequality shows that $y_{\tilde{t}} \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}} A$ and hence condition (c_4) holds. Hence the result is proved.

The result given below is a direct consequence of both Theorem 3.3 and Theorem 3.4.

3.5 Proposition

A is an i-v fuzzy filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ of S if and only if assertions (1), (2) and (3) of Theorem 3.3 and (4) of Theorem 3.4 simultaneously hold for all $x, y \in S$.

Proof. The proof follows from Theorem 3.3 and Theorem 3.4.

Obviously, every ordinary i-v fuzzy left (resp. right) filter is an i-v fuzzy left (resp. right) filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ for all $\tilde{\gamma},\tilde{\delta} \in D[0,1]$. By constructing a counter example we have shown that the converse of the above statement is not true in general.

3.6 Example

The i-v fuzzy subset defined in Example 3.2 is an i-v fuzzy left filter of type $(\in_{[0.3,0.4]}, \in_{[0.3,0.4]} \vee q_{[0.4,0.5]})$ of S but not an ordinary i-v fuzzy left filter of S since $\tilde{F}_A(a) = [0.5, 0.6] > [0.4, 0.5] = \tilde{F}_A(e)$ for $a \leq e$.

Under the stated restriction, the result given below shows that the concepts of i-v fuzzy left filters of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ and ordinary i-v fuzzy left filters coincide.

3.7 Theorem

If for all $x \in S$ we have $\tilde{\gamma} < \tilde{F}_A(x) \leq \tilde{\delta}$, then every i-v fuzzy left filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ of S is an ordinary i-v fuzzy left filter.

Proof. Consider A be an i-v fuzzy left filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ and $a \le b$ (where, $a, b \in S$). By (1) of Theorem (3.3) we have

$\text{r min}\{\tilde{F}_A(a), \tilde{\delta}\} \le \text{r max}\{\tilde{F}_A(b), \tilde{\gamma}\}$.

Hence from the above inequality we have $\tilde{F}_A(a) \le \tilde{F}_A(b)$.

If $x, y \in S$, then by (2) of Theorem (3.3) we have

$\text{r min}\{\tilde{F}_A(x), \tilde{F}_A(y), \tilde{\delta}\} \le \text{r max}\{\tilde{F}_A(xy), \tilde{\gamma}\}$.

From the above inequality it follows that $\text{r min}\{\tilde{F}_A(x), \tilde{F}_A(y)\} \le \tilde{F}_A(xy)$.

And by (3) of Theorem (3.3) we have

$\text{r min}\{\tilde{F}_A(xy), \tilde{\delta}\} \le \text{r max}\{\tilde{F}_A(x), \tilde{\gamma}\}$,

in which it follows that $\tilde{F}_A(xy) \le \tilde{F}_A(x)$.

3.8 Proposition

If for all $x \in S$ we have $\tilde{\gamma} < \tilde{F}_A(x) \le \tilde{\delta}$, then every i-v fuzzy right filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ is an ordinary i-v fuzzy right filter.

Proof. The proof is easy.

3.9 Theorem

The following conditions are equivalent for the i-v fuzzy subset A.

(1) A is an i-v fuzzy left filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$.

(2) For all $\tilde{t} \in D(\tilde{\gamma}, \tilde{\delta}]$ the level subset $U(A; \tilde{t})(\ne \phi)$ of A is an ordinary left filter.

Proof. The proof is straight forward and is omitted.

3.10 Proposition

A is an i-v fuzzy right filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ if and only if For all $\tilde{t} \in D(\tilde{\gamma}, \tilde{\delta}]$ the level subset $U(A; \tilde{t})(\ne \phi)$ of A is an ordinary right filter.

Proof. The proof is straight forward and is omitted.

3.11 Theorem

If A is an i-v fuzzy left filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$, then the set $A_0 := \{x \in S \mid \tilde{F}_A(x) > \tilde{0}$ is a left filter.

Proof. If $a, b \in S$ such that $b \ge a \in A_0$, then $\tilde{F}_A(a) > \tilde{0}$ and by (1) of Theorem (3.3)

$$\begin{aligned}\text{r max}\{\tilde{F}_A(b), \tilde{\gamma}\} &\ge \text{r min}\{\tilde{F}_A(a), \tilde{\delta}\} \\ &> \text{r min}\{\tilde{0}, \tilde{\delta}\} \\ &= \tilde{0}.\end{aligned}$$

showing that $\tilde{F}_A(b) > \tilde{0}$ and hence $b \in A_0$.

If $x, y \in A_0$, then by (2) of Theorem (3.3) we have

$$\begin{aligned}\text{r max}\{\tilde{F}_A(xy), \tilde{\gamma}\} &\ge \text{r min}\{\tilde{F}_A(x), \tilde{F}_A(y), \tilde{\delta}\} \\ &> \text{r min}\{\tilde{0}, \tilde{0}, \tilde{\delta}\} \\ &= \tilde{0}.\end{aligned}$$

This implies $xy \in A_0$.

Finally, if $xy \in A_0$, then by (3) of Theorem (3.3) we have

$$\begin{aligned}\text{r max}\{\tilde{F}_A(x), \tilde{\gamma}\} &\ge \text{r min}\{\tilde{F}_A(xy), \tilde{\delta}\} \\ &> \text{r min}\{\tilde{0}, \tilde{\delta}\} \\ &= \tilde{0}.\end{aligned}$$

Follows that $\tilde{F}_A(x) > \tilde{0}$ and hence $x \in A_0$.

3.12 Proposition

The set $A_0 := \{x \in S \mid \tilde{F}_A(x) > \tilde{0}$ is a right filter whenever A is an i-v fuzzy right filter of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$.

Proof. The proof is easy.

3.13 Proposition

If $\{A_i\}_{i \in I} \ne \phi$ is a collection of i-v $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy left filters of S, then $\bigcap_{i \in I} A_i = \{(x, \bigcap_{i \in I} \tilde{F}_{A_i}(x)) : x \in S\}$ is i-v $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy left filter.

Proof. Consider A_i be i-v $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy left filter of S for each $i \in I$. If $a, b \in S$ such that $a \le b$, then

$$\begin{aligned}\text{r max}\{\bigcap_{i \in I} \tilde{F}_{A_i}(b), \tilde{\gamma}\} &= \bigwedge_{i \in I}\{\text{r max}\{\tilde{F}_{A_i}(b), \tilde{\gamma}\}\} \\ &\ge \bigwedge_{i \in I}\{\text{r min}\{\tilde{F}_{A_i}(a), \tilde{\delta}\}\} \\ &= \text{r min}\{\bigwedge_{i \in I} \tilde{F}_{A_i}(a), \tilde{\delta}\} \\ &= \text{r min}\{\bigcap_{i \in I} \tilde{F}_{A_i}(a), \tilde{\delta}\}.\end{aligned}$$

If $x, y \in S$, then

$$\begin{aligned}\text{r max}\{\bigcap_{i \in I} \tilde{F}_{A_i}(xy), \tilde{\gamma}\} &= \bigwedge_{i \in I}\{\text{r max}\{\tilde{F}_{A_i}(xy), \tilde{\gamma}\}\} \\ &\ge \bigwedge_{i \in I}\{\text{r min}\{\tilde{F}_{A_i}(x), \tilde{F}_{A_i}(y), \tilde{\delta}\}\} \\ &= \text{r min}\{\bigwedge_{i \in I} \tilde{F}_{A_i}(x), \bigwedge_{i \in I} \tilde{F}_{A_i}(y), \tilde{\delta}\} \\ &= \text{r min}\{\bigcap_{i \in I} \tilde{F}_{A_i}(x), \bigcap_{i \in I} \tilde{F}_{A_i}(y), \tilde{\delta}\}.\end{aligned}$$

If $x, y \in S$, then

$$\text{r}\max\{\bigcap_{i\in I}\widetilde{F}_{A_i}(x),\widetilde{\gamma}\} = \wedge_{i\in I}\{\text{r}\max\{\widetilde{F}_{A_i}(x),\widetilde{\gamma}\}\}$$
$$\geq \wedge_{i\in I}\{\text{r}\min\{\widetilde{F}_{A_i}(xy),\widetilde{\delta}\}\}$$
$$= \text{r}\min\{\wedge_{i\in I}\widetilde{F}_{A_i}(xy),\widetilde{\delta}\}$$
$$= \text{r}\min\{\bigcap_{i\in I}\widetilde{F}_{A_i}(xy),\widetilde{\delta}\}.$$

Hence $\bigcap_{i\in I} A_i$ is i-v $(\in_{\widetilde{\gamma}},\in_{\widetilde{\gamma}}\vee q_{\widetilde{\delta}})$-fuzzy left filter of S. Similar, if $\{A_i\}_{i\in I}\neq\phi$ is a collection of i-v $(\in_{\widetilde{\gamma}},\in_{\widetilde{\gamma}}\vee q_{\widetilde{\delta}})$-fuzzy right filters of S, then $\bigcap_{i\in I} A_i$ is i-v $(\in_{\widetilde{\gamma}},\in_{\widetilde{\gamma}}\vee q_{\widetilde{\delta}})$-fuzzy right filter.

4. CONCLUSION

The results obtained in this paper are fundamental for the advanced study on fuzzy ordered semigroups and fuzzy automata. (Zadeh, 1975) introduced the notion of interval fuzzy sets as an extension of fuzzy sets in which the value of the membership degrees are interval of numbers instead of numbers. Since the concept of fuzzy filters of ordered semigroups play an important role in the study of ordered semigroup structure. Therefore, we studied a more general form of interval-valued $(\in,\in\vee\text{q})$-fuzzy left (right) filters in ordered semigroups and established characterisations of an interval-valued $(\in_{\widetilde{\gamma}},\in_{\widetilde{\gamma}}\vee\text{q}_{\widetilde{\delta}})$-fuzzy left (right) filters. Moreover, the obtained results can be applied to semirings, near-rings, hemirings, rings and etc.

REFERENCES:

Biswas. R., (1994). Rosenfeld's fuzzy subgroups with interval-valued membership functions. Fuzzy sets and Systems, Vol. 63 (1): 87-90.

Davvaz. B., (1999). Interval-valued fuzzy subhypergroups. Korean Journal of Computational and Applied Mathematics. Vol. 6 (1): 197-202.

Davvaz. B., and A. Khan, (2012). Generalized fuzzy filters in ordered semigroups. Iranian Journal of Science and Technology, Vol. 1: 77-86.

Davvaz. B., A. Khan., N. H. Sarmin, and H. Khan, (2013). More General Forms of Interval Valued Fuzzy Filters of Ordered Semigroups. International Journal of Fuzzy Systems, Vol. 15 (2): 110-126.

Grattan-Guinness I., (1976). Fuzzy Membership Mapped onto Intervals and Many-Valued Quantities. Mathematical Logic Quarterly, Vol. 22: 149-160.

Jahn. K. U., (1975). Intervall-wertige Mengen. Mathematische Nachrichten, Vol. 68: 115-132.

Kehayopulu. N., and M. Tsingelis, (2002). Fuzzy sets in ordered groupiods. Semigroup Forum, Vol. 65:128-132.

Khan. A., Y. B. Jun., and M. Shabir, (2013). Ordered semigroups characterized by interval valued $(\in,\in\vee\text{q})$-fuzzy bi-ideals. Journal of Intelligent and Fuzzy Systems, Vol. 25 (1): 57-68.

Khan. F. M., A. Khan, and N. H. Sarmin, (2011). Characterizations of ordered semigroups by $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy interior ideals. Lobachevskii Journal of Mathematics, Vol. 32 (4): 278-288.

Khan. F. M., N. H. Sarmin, and H. Khan, (2014). A novel approach toward fuzzy generlaized bi-ideals in ordered semigroups. The Scientific World Journal, Vol. 2014: Article ID 275947, 9 pages http://dx.doi.org/10.1155/2014/275947.

Ma. X., J. Zhan, and Y. B. Jun, (2011). Some kinds of-fuzzy ideals of BCI-algebras. Computers & Mathematics with Applications, Vol. 61 (4):1005-1015.

Murali. V., (2004). Fuzzy points of equivalent fuzzy subsets. Information sciences, Vol. 158: 277-288.

Narayanan. A., and T. Manikantan, (2006). Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semigroups. Journal of Applied Mathematics and Computing, Vol. 20 (1-2): 455-464.

Pao-Ming. P., and L. Ying-Ming, (1980). Fuzzy topology I Neighborhood structure of a fuzzy point and Moore-Smith convergence. Journal of Mathematical Analysis and Applications, Vol. 76: 571-599.

Rosenfeld. A., (1971). Fuzzy groups. Journal of Mathematical Analysis and Applications, Vol. 35: 512-517.

Shabir. M., and I. A. Khan, (2008). Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in ordered semigroups. Mathware & Soft Computing, Vol. 15 (3): 263-272.

Yin. Y., and J. Zhan, (2010). New types of fuzzy filters of BL-algebras. Computers & Mathematics with Applications, Vol. 60 (7): 2115-2125.

Zadeh. L. A., (1975). The concept of a linguistic variable and its application to approximate reasoning-I. Information sciences, Vol. 8 (3): 199-249.

Zadeh. L. A., (1965). Fuzzy sets. Information and control, Vol. 8 (3): 338-353.