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Discrete Generalized Inverted Exponential Distribution

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Abstract: This article presents the Discrete Inverted Exponential Distribution and its Classical Properties. The two-parameter generalized exponential distribution was recently introduced by A.M. Abouammoh and Arwa M. Alshingiti (2009). We presents the three parameter discrete Inverted Exponential distribution is the sum of infinite probability function. Moment estimation, inverse integer moment estimation and maximum likelihood estimation are derived for this infinite probability function.
Keywords: Three parameter Inverted Exponential distribution, Sum of infinite probability function, Moment estimation,

1. INTRODUCTION

The Generalized Inverted Exponential probability distribution has three parameters η, β and x_0. It can be used to represent the failure probability density function (PDF) is given by:

$$f_{GIEP}(x) = \frac{\beta\eta}{(x-x_0)^2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m Exp\left[-\left(\frac{\eta(m+2)}{(x-x_0)}\right)\right], \eta>0, \beta>0, x_0>0, -\infty<x_0<x \quad (1.1)$$

Where $C_{m:\beta-1} = \frac{(\beta-1)!}{m!(\beta-m-1)!}$, β is the shape parameter representing the different pattern of the Generalized Inverted Exponential PDF and is positive and η is a scale parameter representing the characteristic life and the location parameter is x_0. If x_0 =0 then the Generalized Inverted Exponential distribution is said to be two-parameter Generalized Inverted Exponential distribution.

The cumulative distribution function (CDF) of Generalized Inverted Exponential distribution is denoted by $F_{GIEP}(x)$ and is defined as

$$F_{GIEP}(x) = 1-\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m Exp\left(-\left(\frac{\eta(1+m)}{(x-x_0)}\right)\right) \quad (1.2)$$

When the CDF of the Generalized Inverted Exponential distribution has zero value then it represents no failure components by x_0.

The reliability function (RF), denoted by $R_{GIEP}(x)$ is also known as the survivor function and is defined as 1- $F_{GIEP}(x)$

$$R_{GIEP}(x) = \sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m Exp\left(-\left(\frac{\eta(1+m)}{(x-x_0)}\right)\right) \quad (1.3)$$

The hazard function (HF) is also known as instantaneous failure rate denoted by $h_{GEP}(x)$ and is defined as $f_{GEP}(x)/\ R_{GEP}(x)$

$$h_{GIEP}(x)=\frac{\frac{\beta\eta}{(x-x_0)^2}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m Exp\left[-\left(\frac{\eta(m+2)}{(x-x_0)}\right)\right]}{\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m Exp\left(-\left(\frac{\eta(1+m)}{(x-x_0)}\right)\right)} \quad (1.4)$$

The cumulative hazard function (CHF), denoted by $H_{GIEP}(x)$ and is defined as

$$H_{GIEP}(x)=-\ln\left|1-\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m Exp\left(-\left(\frac{\eta(1+m)}{(x-x_0)}\right)\right)\right| \quad (1.5)$$

2. rth Moment Estimation

The rth moment of Generalized Exponential Distribution about origin is given by

$$\mu_r' = \int_{x_0}^{\infty} x^r \frac{\beta\eta}{(x-x_0)^2}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m Exp\left[-\left(\frac{\eta(m+2)}{(x-x_0)}\right)\right]dx \quad (2.1)$$

Put x_0 =0, Then the above reliability model will provide the rth moment of Generalized Exponential Distribution with $\xi_r = \Gamma(1-r)$

$$\mu_r' = \beta\eta^r \sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{(-1)^m \xi_r}{(m+2)^{1-r}}, \quad r=1,2,3,4 \quad (2.2)$$

The special cases of these rth moments of Generalized Exponential Distribution are

$$\mu_1' = \beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m \quad (2.2a) \qquad \mu_2' = \beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)\xi_2 \quad (2.2b)$$

$$\mu_3' = \beta\eta^3\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)^2\xi_3 \quad (2.2c) \qquad \mu_4' = \beta\eta^4\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)^3\xi_4 \quad (2.2d)$$

The variance, skewness and kurtosis measures can now be calculated for the rth moments about mean of Generalized Exponential Distribution using the relations

$$Var(x) = \beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)\xi_2 - \left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)^2$$

$$Skewness(x)=\frac{\beta\eta^3\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)^2\xi_3-3\beta^2\eta^3\left(\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)\left(\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)\xi_2\right)+2\beta^3\eta^3\left(\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)^3}{\left(\beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)\xi_2-\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)^2\right)^{\frac{3}{2}}}$$

$$Kurtosis(x)=\frac{\beta\eta^4\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)^3\xi_4-4\left(\beta\eta^3\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)^2\xi_3\right)\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)+6\left(\beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)\xi_2\right)\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)^2-3\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)^4}{\left(\beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(m+2)\xi_2-\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m\right)^2\right)^2}$$

2. MATERIAL AND METHODS

3. Inverse rth Moment Estimation

The Inverse rth moment estimation of Generalized Exponential Distribution about origin is given by

$$\mu_{r^{-1}}' = \int_{x_0}^{\infty} x^{-r} \frac{\beta\eta}{(x-x_0)^2}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m Exp\left[-\left(\frac{\eta(m+2)}{(x-x_0)}\right)\right]dx \quad (3.1)$$

Put x_0 =0, Then the above reliability model will provide the inverse rth moment of Generalized Exponential Distribution is

$$\mu'_{r^{-1}} = \beta\eta^{-r}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\,\frac{(-1)^m\,\Gamma(1+r)}{(m+2)^{1+r}}\,,\; r = 1,2,3,4 \qquad (3.2)$$

The special cases of these inverse rth moments of Generalized Exponential Distribution are

$$\mu'_{1^{-1}} = \beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\,\frac{(-1)^m}{(m+2)^2} \qquad (3.2a)$$

$$\mu'_{2^{-1}} = 2\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\,\frac{(-1)^m}{(m+2)^2} \qquad (3.2b)$$

$$\mu'_{3^{-1}} = 6\beta\eta^{-3}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\,\frac{(-1)^m}{(m+2)^3} \qquad (3.2c)$$

$$\mu'_{4^{-1}} = 24\beta\eta^{-4}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\,\frac{(-1)^m}{(m+2)^4} \qquad (3.2d)$$

The variance, skewness and kurtosis measures can now be calculated for the Inverse rth moments about mean of Generalized Exponential Distribution using the relations

$$Var(x^{-1}) = 2\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\,\frac{(-1)^m}{(m+2)^2} - \left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\,\frac{(-1)^m}{(m+2)^2}\right)^2 \qquad (3.3)$$

$$Skewness(x^{-1}) = \frac{6\beta\eta^{-3}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^3} - 3\left(2\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^2}\right)\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right) + 2\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right)^3}{\left(2\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^2} - \left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right)^2\right)^{\frac{3}{2}}} \qquad (3.4)$$

$$Kurtosis(x^{-1}) = \frac{24\beta\eta^{-4}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^4} - 4\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right)\left(6\beta\eta^{-3}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^3}\right) + 6\left(2\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^2}\right)\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right)^2 - 3\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right)^4}{\left(2\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\frac{(-1)^m}{(m+2)^2} - \left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m}{(m+2)^2}\right)^2\right)^2} \qquad (3.5)$$

3. <u>RESULTS</u>

4. L-Moments Estimation

By Hosking (1990), L-moments are expectation of certain linear combinations of order Statistics. Hosking has defined the L-moments of X to be the quantities.

$$\lambda_r = r^{-1}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}E\left(X_{r-k:r}\right)\quad , r = 1, 2, 3, 4 \qquad (4.1)$$

The L in “L-moments” emphasizes that λ_r is a linear function of the expected order statistics. The expectation of an order Statistics has been written as (Hosking (1990))

$$EX_{j:r} = \frac{r!}{(j-1)!\,(r-j)!}\int x(F)\{F(x)\}^{j-1}\{1-F(x)\}^{r-j}\,dF(x). \qquad (4.2)$$

The first few L-moments, λ_r of random variable”X”, as defined by Hosking (1990) are given below:

$$\lambda_1 = E(X) = \int_0^1 x(F)\,dF \qquad (4.2a)$$

$$\lambda_2 = \frac{1}{2}E\left(X_{2:2} - X_{1:2}\right) = \int_0^1 x(F)(2F-1)\,dF \qquad (4.2b)$$

$$\lambda_3 = \frac{1}{3}E\left(X_{3:3} - 2X_{2:3} + X_{1:3}\right) = \int_0^1 x(F)\left(6F^2\;6F+1\right)dF \qquad (4.2c)$$

$$\lambda_4 = \frac{1}{4}E\left(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}\right) = \int_0^1 x(F)\left(20F^3 - 30F^2 + 12F - 1\right)dF \qquad (4.2d)$$

Where $X_{k:n}$ is an order Statistics, the k[th] smallest of a sample of size "n" drawn from the distribution of X and x(F) is a quantile function of real-valued random variable X. The "L" in L-moment emphasized that λ_r is a linear function of expected order Statistics. Measures of skewness and kurtosis, based on L-moments, are respectively.

L-skewness = $\tau_3 = \lambda_3 / \lambda_2$ (4.3) L-Kurtosis = $\tau_4 = \lambda_4 / \lambda_2$ (4.4)

By solving equation (4.2a), (4.2b), (4.2c) and (4.2d) we get the L moments of discrete Inverted exponential distribution

$$\lambda_1 = \alpha\beta \sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x \quad (4.4) \qquad \lambda_2 = \alpha\beta \left[\sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x - 2 \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x \right] \quad (4.5)$$

$$\lambda_3 = 6\alpha\beta \sum_{x=1}^{3\alpha-1} C_{x:3\alpha-1} (-1)^x - 6\alpha\beta \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x + \alpha\beta \sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x \quad (4.6)$$

$$\lambda_4 = \alpha\beta \sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x - 12\alpha\beta \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x + 30\alpha\beta \sum_{x=1}^{3\alpha-1} C_{x:3\alpha-1} (-1)^x - 20\alpha\beta \sum_{x=1}^{4\alpha-1} C_{x:4\alpha-1} (-1)^x \quad (4.7)$$

The L-Coefficient of Variation of discrete Inverted exponential distribution is in eq (4.5)

$$C.V = \sqrt{\frac{\alpha\beta \left[\sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x - 2 \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x \right]}{\alpha\beta \sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x}} \quad (4.8)$$

By solving equation (4.3) we get the L-Coefficient of Skewness of discrete Inverted exponential distribution

$$\tau_1 = \frac{6\alpha\beta \sum_{x=1}^{3\alpha-1} C_{x:3\alpha-1} (-1)^x - 6\alpha\beta \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x + \alpha\beta \sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x}{\alpha\beta \left[\sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x - 2 \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x \right]} \quad (4.9)$$

By solving equation (4.3) we get the L-Coefficient of Kurtosis of discrete Inverted exponential distribution

$$\tau_2 = \frac{\alpha\beta \sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x - 12\alpha\beta \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x + 30\alpha\beta \sum_{x=1}^{3\alpha-1} C_{x:3\alpha-1} (-1)^x - 20\alpha\beta \sum_{x=1}^{4\alpha-1} C_{x:4\alpha-1} (-1)^x}{\alpha\beta \left[\sum_{x=1}^{\alpha-1} C_{x:\alpha-1} (-1)^x - 2 \sum_{x=1}^{2\alpha-1} C_{x:2\alpha-1} (-1)^x \right]} \quad (4.10)$$

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