Sindh Univ. Res. Jour. (Sci. Ser.) Vol. 41 (1) 87-94 (2009)

Sindh Univ. Res. Jour.

SURJ

MODIFIED INFERENCE ABOUT THE SCALE PARAMETER OF THE INVERSE WEIBULL DISTRIBUTION BY USING TYPE II CENSORED SAMPLE

M. Aleem and F. Jamal

Department of statistics, The Islamia University of Bahawalpur Pakistan

E-Mail. draleemiub@hotmail.com

(Received 11th April 2009 and Revised 28th May 2009)

Abstract

Suresh (2004) used Taylor expansion series to linearize the intractable term in likelihood equation. In this Paper a simple approximation has been proposed for intractable term , to estimate scale parameter "θ" keeping shape parameter "m" fixed of two parameter Inverse Weibull distribution from doubly type ll censored sample and studying the effect of censored sample in terms of asymptotic variances and Mean square error .

Keywords: Type II censored sample, Modified Maximum Likelihood Estimator, Mean Square Error, Bias, Order Statistics, Inverse weibull distribution.

1. Introduction

The probability density function of inverse Weibull distribution is given as:

$$f(y) = \frac{m}{\theta}\frac{1}{y^{m+1}}\exp\left(-\frac{1}{\theta\, y^{m}}\right), \; y > 0,\; m,\; \theta > 0. \qquad (1.1)$$

= 0 other wise .

and the corresponding distribution function is:

$$F(y) = \exp\left(-\frac{1}{\theta\, y^{m}}\right) \qquad (1.2)$$

Where "m" is shape and "θ" is scale parameter

2. The Modified Maximum Likelihood Estimator (Mmle) Of The Scale Parameter Θ Of The Inverse Weibull Distribution

For the doubly type ll censored sample with r samples censored on the left and s samples censored on the right .The likelihood function is given as

$$L = \frac{n!}{r!s!}\left[F(Y_{r+1})\right]^{r}\left[1-F(Y_{n-s})\right]^{s}\prod_{i=r+1}^{n-s} f(y_i) \qquad (2.1)$$

Where $r = [nq_1]+1$ and $s = [nq_2]+1$, q_1 is the proportion of left censored sample and q_2 is the proportion of right censored sample .By using (1.2) in (2.1) we get

$$L = \frac{n!}{r!s!}\left[\left(\exp\left(-\frac{1}{\theta\, {y_{r+1}}^{m}}\right)\right)\right]^{r}\left[1-\exp\left(-\frac{1}{\theta\, {y_{n-s}}^{m}}\right)\right]^{s}\prod_{i=r+1}^{n-s}\left[\frac{m}{\theta}\frac{1}{{y_i}^{m+1}}\exp\left(-\frac{1}{\theta\, {y_i}^{m}}\right)\right]$$

The first derivatives of the log-likelihood function with respect to θ is given by:

$$\frac{\partial \ln L}{\partial \theta} = r\left(\frac{1}{\theta^2\ {y_{r+1}}^m}\right) + \frac{s}{\theta^2\ {y_{n-s}}^m}\left[-\frac{\exp\left(-\frac{1}{\theta\ {y_{n-s}}^m}\right)}{1-\exp\left(-\frac{1}{\theta\ {y_{n-s}}^m}\right)}\right] - \frac{(n-s-r)}{\theta} + \frac{1}{\theta^2}\sum_{i=r+1}^{n-s}\frac{1}{{y^m}_i} \overset{(set)}{=} 0 \qquad (2.2)$$

By using $\exp\left(z^{-1}\right) = \frac{1+\frac{1}{2z}}{1-\frac{1}{2z}}$ for intractable term in (2.2), see Suresh (2004) .Solving (2.2) for θ,the modified maximum likelihood estimator of θ is given as

$$\theta^{\wedge} = \frac{\frac{r}{{y_{r+1}}^m} + \frac{s}{2{y^m}_{n-s}} + \sum_{i=r+1}^{n-s}\frac{1}{{y^m}_i}}{n-r} \qquad (2.3)$$

3. Asymptotic Variance and Bias Of $\theta^{\wedge}$

By using Glivenko-Cantelli lemma given by john and Chen (2005b),
We have

$p_1 = G^{-1}(q_1)$ and $p_2 = G^{-1}(1-q_2)$ So as $n \to \infty$

$z_{r+1} = G^{-1}(q_1)$ and $z_{n-s} = G^{-1}(1-q_2)$

z_{r+1} can be evaluated by using following relation

$$G(z_{r+1}) = q_1$$

$$\int_0^{z_{r+1}} f(z)dz = q_1$$

$$z_{r+1} = -\frac{1}{\ln q1} \qquad (3.1)$$

and

$$G(z_{n-s}) = 1-q_2$$

$$\int_{z_{n-s}}^{\infty} f(z)dz = q_2$$

$$z_{n-s} = -\frac{1}{\ln(1-q_2)} \qquad (3.2)$$

and

$$\lim_{n\to\infty} E\left\{\frac{1}{n}\sum_{i=r+1}^{n-s}\frac{1}{z_i}\right\} = \int_{z_{r+1}}^{z_{n-s}} \frac{1}{z} f(z)dz = \int_{z_{r+1}}^{z_{n-s}} \frac{1}{z^3}\exp\left(-\frac{1}{z}\right)dz$$

$$= 1-\ln(1-q_2)-q_2+q_2\ln(1-q_2)-q_1+q_1\ln q_1 \qquad (3.3)$$

Put $z_{n-s} = \theta y^m{}_{n-s}$ for intractable term in equation (2.2)

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta^2 y^m{}_{r+1}} + \frac{s}{\theta^2 y^m{}_{n-s}}\left(\frac{-\exp\left(-\frac{1}{z_{n-s}}\right)}{1-\exp\left(-\frac{1}{z_{n-s}}\right)}\right) - \frac{(n-s-r)}{\theta} + \frac{1}{\theta^2}\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i}$$

Again use modification by Suresh (2004) to solve intractable term of the above equation we get

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta^2 y^m{}_{r+1}} + \frac{s}{2\theta^2 y^m{}_{n-s}} - \frac{s}{\theta} - \frac{(n-s-r)}{\theta} + \frac{\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i}}{\theta^2}$$

Partially differentiate the above equation w.r.t θ

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2r}{\theta^3 y^m{}_{r+1}} - \frac{2s}{2\theta^3 y^m{}_{n-s}} + \frac{s}{\theta^2} + \frac{(n-s-r)}{\theta^2} - \frac{2\sum_{i=r+1}^{n-s}\frac{1}{y^m{}_i}}{\theta^3}$$

Putting $z_{r+1} = y^m{}_{r+1}\theta,\ z_{n-s} = y^m{}_{n-s}\theta,\ z_i = y^m{}_i\theta$,we have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{2r}{z_{r+1}\theta^2} - \frac{s}{z_{n-s}\theta^2} + \frac{s}{\theta^2} + \frac{(n-s-r)}{\theta^2} - \frac{2\sum_{i=r+1}^{n-s}\frac{1}{z_i}}{\theta^2} \qquad (3.4)$$

By using equation (3.1) and (3.2) we obtain as:

$$-E\left(\frac{\partial^2 \ln L}{\partial \theta^2}\right) = \frac{n}{\theta^2}\left(-2q_1\ln q_1 - q_2\ln(1-q_2) - 1 + q_1 + 2E\left(\frac{1}{n}\sum_{i=r+1}^{n-s}\frac{1}{z_i}\right)\right)$$

From equation (3.3)

$$= \frac{n}{\theta^2}\left[-2q_1\ln q_1 - q_2\ln(1-q_2) - 1 + q_1 + 2 - 2\ln(1-q_2) - 2q_2 + 2q_2\ln(1-q_2) - 2q_1 + 2q_1\ln q_1\right]$$

So, Asymptotic variance is given as

$$\operatorname{var}(\hat{\theta}) = \frac{\theta^2}{n[1-q_1-2q_2-2\ln(1-q_2)+q_2\ln(1-q_2)]} \qquad (3.5)$$

Where $\therefore q_1 = \frac{r}{n} \therefore q_2 = \frac{s}{n}$

From (3.5) we observe as $n \to \infty, \operatorname{var}(\hat{\theta}) \to 0$

Now put $z_{r+1} = y^m{}_{r+1}\theta, z_{n-s} = y^m{}_{n-s}\theta$ and $z_i = y^m{}_i\theta$ in the equation (2.3), we get

$$\theta^{\wedge} = \frac{\dfrac{\theta r}{z_{r+1}} + \dfrac{s\theta}{2 z_{n-s}} + \theta \sum_{i=r+1}^{n-s} \dfrac{1}{z_i}}{n-r}$$

Apply expectation on both sides we have

$$Bias = E\left(\theta^{\wedge}\right) - \theta = \frac{\theta\left(\frac{1}{2} q_2 \ln(1-q_2) - \ln(1-q_1) - q_2\right)}{1-q_1} \qquad (3.6)$$

From (3.6) we observe as sample size increases, The Bias decreases i.e. $Bias \to 0$ for large sample size

4. Estimation of the Mean Square Error

From Aleem (2004) for moments of order statistics from inverse weibull distribution, we have

$\mu_{r:n}^{(k-m)} = E\left(y_{r:n}^{k-m}\right)$ then

$$\mu_{r:n}^{(k-m)} = \frac{C_{r:n}\Gamma\left(2 - \frac{k}{m}\right)}{\theta^{\frac{k}{m}-1}} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j (a_j)^{\frac{k}{m}-2} \qquad (4.1)$$

Where $C_{r:n} = \dfrac{n!}{(r-1)!\,(n-r)!}$

Putting k=0 in (4.1) we obtain

$$\mu_{r:n}^{(-m)} = E\left(y_{r:n}^{-m}\right)$$

$$E\left(\frac{1}{y_{r:n}^{m}}\right) = \theta \frac{n!}{(r-1)!\,(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left(\frac{1}{r+j}\right)^2 \qquad (4.2)$$

Where $\alpha_r = \dfrac{n!}{(r-1)!\,(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left(\dfrac{1}{r+j}\right)^2$

And by putting k =-m in (4.1) ,we have

$$\mu_{r:n}^{(-m-m)} = E\left(y_{r:n}^{-2m}\right) = E\left(\frac{1}{y_{r:n}^{m}}\right)^2$$

$$E\left(\frac{1}{y_{r:n}^{m}}\right)^2 = 2\theta^2 \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left(\frac{1}{r+j}\right)^3 \qquad (4.3)$$

By using (4.2 and (4.3),we have)

$$\operatorname{var}\left(\frac{1}{y^{m}{}_{r}}\right) = \theta^2 \beta_r \qquad (4.4)$$

Where $\beta_r = 2\dfrac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left(\dfrac{1}{r+j}\right)^3 - (\alpha_r)^2$

Applying expectation on equation (2.3) and then by using equation (4.2), we obtain

$$Bias(\theta^{\wedge}) = E(\theta^{\wedge}) - \theta = \frac{\theta\left(q_1\alpha_{r+1} + \frac{q_2}{2}\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1\right)}{1-q_1} \quad (4.5)$$

Where $\alpha = \sum_{i=r+1}^{n-s}\alpha_i$

Now applying variance on equation (2.3),then by using equation (4.4) we obtained

$$Var(\theta^{\wedge}) = \frac{\theta^2\left(q^2{}_1\beta_{r+1} + \frac{q^2{}_2}{4}\beta_{n-s} + \frac{\beta}{n^2}\right)}{(1-q_1)^2} \quad (4.6)$$

Where $\beta = \sum_{i=r+1}^{n-s}\beta_i$

By using equation (4.5) and (4.6),we have

$$MSE(\theta^{\wedge}) = \frac{\theta^2}{(1-q_1)^2}\left(\left(q_1\alpha_{r+1} + \frac{q_2}{2}\alpha_{n-s} + \frac{\alpha}{n} - 1 + q_1\right)^2 + \left(q^2{}_1\beta_{r+1} + \frac{q^2{}_2}{4}\beta_{n-s}\frac{\beta}{n^2}\right)\right) \quad (4.7)$$

5. Comparison Of Censored Sample Verses Complete Sample

$\theta^{\wedge} = \frac{\sum_{i=1}^{n}\frac{1}{y^m{}_i}}{n}$ is estimator of complete sample with

$$MSE(\theta^{\wedge}) = \frac{\theta^2}{n^2}\left(\left(\sum_{i=1}^{n}\alpha_i - n\right)^2 + \left(\sum_{i=1}^{n}\beta_i\right)\right) \quad (5.1)$$

Reduction in efficiency = (MSE of censored sample / MSE of complete sample)-1
Which can be obtained from the equation (4.7) and (5.1) (5.2)

Table-5.1: For the asymptotic variance of MML estimates from doubly censored sample in term of θ^2 for n=10, 20, 30, 50,100from the equation (3.1)

q1	q2	n=10	n=20	n=30	n=50	n=100
		var $(\theta)/\theta^2$	var $(\theta)/\theta^2$	var $(\theta)/\theta^2$	var $(\theta)/\theta^2$	var $(\theta)/\theta^2$
0	0	0.1	0.05	0.0333	0.02	0.01
0	0.1	0.1	0.05	0.0333	0.02	0.01
0	0.2	0.0998	0.0499	0.0333	0.02	0.01
0	0.3	0.0994	0.0497	0.0331	0.0199	0.0099
0	0.4	0.0983	0.0491	0.0328	0.0197	0.0098
0	0.5	0.0962	0.0481	0.0321	0.0192	0.0096
0	0.6	0.0924	0.0462	0.0308	0.0185	0.0092
0.1	0	0.1111	0.0556	0.037	0.0222	0.0111

0.1	0.1	0.1111	0.0555	0.037	0.0222	0.0111
0.1	0.2	0.1109	0.0555	0.037	0.0222	0.0111
0.1	0.3	0.1103	0.0552	0.0368	0.0221	0.011
0.1	0.4	0.109	0.0545	0.0363	0.0218	0.0109
0.1	0.5	0.1064	0.0532	0.0355	0.0213	0.0106
0.2	0	0.125	0.0625	0.0417	0.025	0.0125
0.2	0.1	0.125	0.0625	0.0417	0.025	0.0125
0.2	0.2	0.1247	0.0624	0.0416	0.0249	0.0125
0.2	0.3	0.124	0.062	0.0413	0.0248	0.0124
0.2	0.4	0.1224	0.0612	0.0408	0.0245	0.0122
0.3	0	0.1429	0.0714	0.0476	0.0286	0.0143
0.3	0.1	0.1428	0.0714	0.0476	0.0286	0.0143
0.3	0.2	0.1425	0.0713	0.0475	0.0285	0.0143
0.3	0.3	0.1416	0.0708	0.0472	0.0283	0.0142

Table-5.2: For Bias of MML estimates from doubly censored sample in term of θ for n=10,20,30 from the equation (4.5)

q1	q2	n=10	n=20	n=30
		$Bias(\theta)/\theta$	$Bias(\theta)/\theta$	$Bias(\theta)/\theta$
0	0	-4.11E-15	-6.82E-11	-8.86E-07
0	0.1	0.000555	0.000277	0.000204
0	0.2	0.0025	0.0016	0.0013
0	0.3	0.0071	0.005	0.0043
0	0.4	0.0165	0.0123	0.011
0	0.5	0.0342	0.0266	0.0242
0	0.6	0.0669	0.0534	0.0493
0.1	0	-6.38E-15	-8.05E-11	-9.88E-07
0.1	0.1	0.000617	0.000308	0.000227
0.1	0.2	0.0028	0.0017	0.0014
0.1	0.3	0.0079	0.0055	0.0048
0.1	0.4	0.0183	0.0137	0.0122
0.1	0.5	0.038	0.0295	0.0269
0.2	0	-1.04E-15	-3.20E-11	-2.18E-06
0.2	0.1	0.000694	0.000347	0.000255
0.2	0.2	0.0031	0.0019	0.0016
0.2	0.3	0.0089	0.0062	0.0054
0.2	0.4	0.0206	0.0154	0.0138
0.3	0	-1.85E-14	3.86E-10	2.95E-05
0.3	0.1	0.000793	0.000396	0.000324
0.3	0.2	0.0036	0.0022	0.0019
0.3	0.3	0.0102	0.0071	0.0062

Table -5.3: For MSE of MML estimates from doubly censored sample in term of θ^2 for n=10,20,30 from the equation (4.7)

q1	q2	n=10 $MSE(\theta^2)/\theta^2$	n=20 $MSE(\theta^2)/\theta^2$	n=30 $MSE(\theta^2)/\theta^2$
0	0	0.0293	0.009	0.0044
0	0.1	0.0292	0.009	0.0044
0	0.2	0.0294	0.0091	0.0045
0	0.3	0.03	0.0094	0.0047
0	0.4	0.0317	0.0104	0.0054
0	0.5	0.0362	0.0127	0.0071
0	0.6	0.0473	0.0186	0.0114
0.1	0	0.0238	0.0086	0.005
0.1	0.1	0.0238	0.0086	0.005
0.1	0.2	0.0239	0.0087	0.0051
0.1	0.3	0.0246	0.0091	0.0054
0.1	0.4	0.0268	0.0103	0.0062
0.1	0.5	0.0323	0.0132	0.0083
0.2	0	0.0317	0.014	0.009
0.2	0.1	0.0316	0.014	0.009
0.2	0.2	0.0318	0.0141	0.0091
0.2	0.3	0.0327	0.0147	0.0095
0.2	0.4	0.0355	0.0162	0.0105
0.3	0	0.0455	0.0219	0.0144
0.3	0.1	0.0454	0.0219	0.0144
0.3	0.2	0.0456	0.0221	0.0146
0.3	0.3	0.0468	0.0228	0.0151

R=Reduction in efficiency = (MSE of censored sample / MSE of complete sample) -1

Table -5.4: For Comparison of censored sample compare to complete sample for n=10,20,30 as from the equation 5.2

q1	q2	n=10 $R(\theta)-1$	n=20 $R(\theta)-1$	n=30 $R(\theta)-1$
0	0	0	0	0
0	0.1	-0.0015	0.00017	0.001
0	0.2	0.0021	0.01	0.0158
0	0.3	0.0226	0.0482	0.0699
0	0.4	0.083	0.1529	0.2173
0	0.5	0.2358	0.4135	0.5911
0	0.6	0.6163	1.0697	1.561
0.1	0	-0.1869	-0.0426	0.1335
0.1	0.1	-0.1888	-0.0424	0.1348
0.1	0.2	-0.1843	-0.0303	0.1529
0.1	0.3	-0.1591	0.0169	0.2197
0.1	0.4	-0.0844	0.1462	0.4018
0.1	0.5	0.1042	0.4679	0.8633
0.2	0	0.0821	0.557	1.0213
0.2	0.1	0.0798	0.5572	1.023
0.2	0.2	0.0855	0.5726	1.046
0.2	0.3	0.1174	0.6323	1.1305
0.2	0.4	0.2119	0.7959	1.3609
0.3	0	0.5521	1.4341	2.2517
0.3	0.1	0.5491	1.4344	2.2538
0.3	0.2	0.5565	1.4544	2.2839
0.3	0.3	0.5982	1.5324	2.3943

6. Discussion and Conclusion

1- From table **(5.1)** the following conclusions are made.

- As the sample size increase, the variances are decrease with the same proportion of left and right –censored sample.
- For any sample size there are three conditions for q_1 and q_2 .
  - $q_1 > q_2$
  - $q_1 < q_2$
  - $q_1 = q_2$

for the 1st two condition it is noticed that the variances are increased when $q_1 > q_2$ as compared to $q_1 < q_2$ for the same sample size .So we may conclude that the large number of left censored samples is caused to increase variances.

- The variance of scale parameter does not depend on total proportion, as q_1 increases, the variance also increases .
- For $q_1=q_2=0$. no censoring scheme is involved in other words there is no missing element in the sample, the asymptotic variance of MML estimator is minimum as compare to when $q_1 > q_2$

2- From table **(5.2)** the following conclusions are made.

- As the sample size increases, the term of bias decreases with same proportion of left and right-censored sample.
- As the total proportion of censored sample increases, the bias also increases.

3- From table **(5.3)** the following conclusions are made.

- For $q_1=q_2=0$. no censoring scheme is involved in other words there is no missing element in the sample , the Mean square error of MML estimator is minimum.

The Mean square error of scale parameter does not depend on total proportion , as q_1 increases, the Mean square error also increases .

4- Comparison of results table **(5.1)** and table **(5.3)**

The results of both tables **(5.1 and 5.3)** give the variation of estimator. Comparing both tables the following conclusions are made.

- The asymptotic variance can be use for any sample of size n , either small or large. But the Mean square error obtained by using theory of ordered statistics can be use only for small sample size of n.
- The Mean square error is minimum as compared to asymptotic variance

5-From table **(5.4)** the reduction in efficiency has been calculated, the following conclusion are made

- When $q_1=q_2=0$. no censoring scheme is involved in other words there is no missing element in the sample, no effect seen on efficiency.
- As the sample size increases the reduction in efficiency also increases with the same proportion of left and right censored sample.
- As the total proportion of censored sample increases , the reduction in efficiency also increases.

References

Akhter, S. (2006) a study on parameter estimation of exponential distribution for type-ll censored sample M. Phil thesis department of statistics G.C.U Lahore Pakistan.

Aleem. M. (2004) classical properties, ratio, product and inverse moments of order statistics and characterization from inverse weibull distribution. J. Of stats. Vol. Xi (1), 17- 24.

Jamal, F. (2008) on the parameters estimation of the inverse weibull distribution M. Phil thesis department of statistics The Islamia University of Bahawalpur, Pakistan.

John, T. T. and P. Chen (2003) Selection of the Best using Censored Data from (log-) location - Scale Populations. Preprint submitted to journal of Statistical Planning and Inference 24.

Suresh, R. P. (2004) Estimation of location and scale parameters in a two parameter exponential distribution from a censored sample , statistical methods , 6(1), 82-89.